Team Formation in Firms: Matching with Search and Separation

Abstract

We adopt dynamic matching theory to model team formation in organizations. Workers and managers are heterogeneous in quality and select each other in order to form teams. Searching for a teammate is costly. Our primary theoretical innovation is to allow agents to search while matched and separate if they find a better teammate. If one side of the market has the ability to separate, we achieve positive assortative matching: the best managers match with the best workers. If both sides can separate, weak conditions on payoffs and search technology ensure that the steady state converges to perfect positively assortative matching.

1 Introduction

Research on organization design in accounting, economics, and finance has focused almost exclusively on agency theory, where firms seek to mitigate underlying moral hazard and adverse selection problems. While this vast and still growing literature has led to deep insights into optimal contracts and organization, there is a competing corpus of theory that is useful to explain internal firm structure. Namely, the theory of sorting and matching asks not how to obtain first-best effort provision, but how two sides of a heterogeneous market match with one another.\(^1\)

Two sided matching theory abstracts away from agency problem, and instead seeks to understand who matches with whom in both social and productive settings. Because of this, matching theory is well suited to understand the decentralized formation of teams in organizations. Of course, there already exists a large incentive-based literature on teams and organizations, mostly focusing on free-riding and monitoring issues (Baldenius et al. [2002]; Harris and Raviv [2002]; Holmstrom

But these papers take the team as given, rather than examining how teams form. And while the majority of matching papers operate in market environments outside of firms (external labor market matching and marriage models), there is a growing literature of importing market, and market like structures, into firms (see Baiman et al. [2007] on auctions within firms.)

We extend the two-sided search and matching model to include the possibility of separation. Specifically, managers and workers on both sides of the market are heterogeneous in their quality, the payoff function from their match exhibits complementarity (so a manager’s or worker’s payoff increases in the quality of its teammate), and it is costly to search for matches. Utility is nontransferable, so wages are not available to equilibrate between matches. The nontransferable utility is particularly appropriate in firms where intramatch transfers are rare, and where agents “buy” each other with their own quality rather than with dollars. We add to this standard model that agents can search while matched, separate and rematch if they find a better teammate. Our main result is that the ability to separate ex-post improves matching ex-ante.

The logic is simple. Without separation, agents are locked into a match forever. Because of this, agents are naturally reluctant to match early because they are waiting for the perfect match. But searching is costly both in time and effort, so agents trade off the desire to be in the right match against the deadweight loss of search. As previous papers in this literature have shown, this results in an equilibrium where agents match in blocks: each agent within an interval will match to any other agent within that interval. However, this is inefficient, not only because of the deadweight loss of search, but also because it does not optimize the joint payoff function. Under complementarity in production, the efficient outcome is when a manager of quality $x$ matches a worker of quality $x$. But when agents are locked into their matches forever, they match in blocks rather than directly to agents of the same quality.

Allowing separation eliminates this. Now there is no need to match in blocks because each party has the ability to upgrade later. So, agents match early, and if they find a better match, they separate and rematch. Over time, the resulting equilibrium converges to the efficient equilibrium of each agent matching to their similar qualities (positive assortative matching). Allowing separation gives each party an exit option, and that option has value because it can dissolve inefficient matches and replace them with more efficient ones. While there may be winners and losers in the dissolution of individual matches, society as a whole is better off because each agent is eventually matched with their equal.

To see concretely how separation improves matching and increases efficiency, first consider the world without separation. Because matches are permanent and waiting is costly, agents match in blocks. They effectively choose to match with an entire range of teammates of similar (but not identical) quality. For example, it is possible that within a single block an $x$-manager will agree
to match with a worker of quality \( y \) (\( y \)-worker) where \( y > x \). But if both parties can search while matched, then they may meet a manager or worker closer to their own quality. For example, the \( y \)-worker may meet a \( z \)-manager where \( z > x \). In that case, the \( y \)-worker dissolves his match with the \( x \)-manager, and matches with the better \( z \)-manager. While the \( x \)-manager is (temporarily) worse off after the separation, the \( y \)-worker is certainly better off with a better match. Because of complementarity in the payoffs, the benefit to the \( y \)-worker exceeds the loss to the \( x \)-manager. Eventually the \( x \)-manager will find a worker in his own league. Separation increases efficiency by improving the matching process. Without separation, the cost of waiting and the permanence of the match forces agents to match in blocks, which effectively causes them to match to the “wrong” agent. Allowing separation leads agents eventually to their “right” match, namely, their agent of same quality. This maximizes total surplus.

Before we turn to our model, let us first take a closer look at the existing theoretical literature on matching. This literature began with a seminal paper by Gale and Shapley [1962], in which they demonstrated that certain match markets always allow “stable” matchings, in which no agent prefers another to their currently matched agent; this equilibrium concept is equivalent to the core in this situation (as noted by, e.g., Shapley and Shubik [1972]).\(^2\) One of the key insights in this class of matching models is due to Becker [1973]: when the payoff function is supermodular (i.e., there are complementarities in production), the only efficient allocation, and hence the unique core allocation, is the one where each agent matches only to his or her own type. This matching pattern is called \textit{perfect positively assortative matching} (perfect PAM).

Models such as these typically examine the core, or perhaps competitive equilibria, of matching markets. To achieve these outcomes, the models tend to rely on a massive clearinghouse, which solicits preference reports from every agent on each side and constructs a matching according to some algorithm. The process is often framed as dynamic, but this is usually just for illumination of the algorithm itself. Even if no clearinghouse exists, each agent is assumed to be able to meet all potential teammates without losing any time or exerting any effort. In sum, these early models depend crucially on the assumption of no search costs.

This has been explored in a large number of papers in the 1990s, starting with the seminal paper of McNamara and Collins [1990] and culminating with the insightful papers by Shimer and Smith [2000, with transferable utility] and Smith [2006, with nontransferable utility]. Like our paper, these papers explore the steady state of the system, similar to the standard approach in evolutionary game theory, where the pivotal concept is that of an evolutionarily stable strategy.

\(^2\)Labor markets have been studied by Crawford and Knoer [1981]; Demange and Gale [1985]; Kelso and Crawford [1982]; Shapley and Shubik [1972], often with the additional element of a monetary transfer between workers and workers (namely, a salary). Roth and Sotomayor [1990] constructed a thorough discussion of two-sided matching theory, with particular attention to the empirical case of the National Resident Matching Program (NRMP).
or ESS, as introduced by Smith and Price [1973]. When utility is nontransferable, papers in this literature commonly find the phenomenon of block segregation discussed earlier: agents on each side of the market separate themselves into fixed bands or intervals of quality, such that an agent in a given band can be matched with any agent from the corresponding band on the other side, but to no agent from any other bands. In particular, this implies that a small change in an agent’s quality results either in no change or a discontinuous jump in the set of possible teammates for that agent. These papers show that the phenomenon is robust under continuous and discrete time, proportional discounting of the future and constant search costs, and various forms of the payoff function. Smith [2006] was the first to use a general payoff function to show that block segregation arises with any payoff function that is multiplicatively separable in the two teammates’ payoffs and shows why this phenomenon occurs.

Block segregation has always raised criticism because it simply does not seem to fit with our sense of reality; Smith [2006, page 1134] writes that “there are no documented cases of block segregation.” In addition, a discontinuous equilibrium in an entirely continuous model is in and of itself suspicious. It should also be noted that when payoffs are supermodular, block segregation is clearly inefficient. Both the implausibility and the inefficiency of the phenomenon prompt us to look for conditions that would eliminate this outcome.

Block segregation (as well as other, less extreme, inefficiencies in costly-search models) arises in large part from the need of agents to weigh the benefits of matching immediately against the opportunity cost of giving up the search for a better match in the future. It is therefore natural to suspect that block segregation could be ruled out if agreeing to match with an agent did not entail giving up search for better matches in the future. We find that allowing matched agents to continue searching for upgrades eliminates block segregation. Even more interestingly, when payoffs are supermodular, allowing both sides of the market to search for upgrades can often almost completely eliminate search inefficiencies in steady state.

Even when block segregation does not obtain, the set of potential agents for each agent in a model with costly search is a non-singleton set. Thus, the idea of perfect assortativeness from the frictionless model is of no direct use. To extend the idea of “like matches to like” to such models, Shimer and Smith [2000] introduce the concept of setwise positively assortative matching (setwise PAM): in setwise PAM, higher agents match to higher sets of agents. Smith [2006] shows that a sufficient condition for matching to be strictly assortative setwise is that the payoff function be strictly log-supermodular (note that strict supermodularity is not sufficient). When payoffs are multiplica-

---

3 A full dynamic analysis of the system is unfortunately complex. For an idea of what happens out of steady state, see the biology paper by Alpern and Reyniers [2005] and the ongoing project by Smith [2002].

4 This peculiar discontinuity in the otherwise continuous matching model was first discovered by McNamara and Collins [1990] and was subsequently revisited by authors such as Bloch and Ryder [2000]; Burdett and Coles [1997]; Chade [2001]; Eeckhout [1999]; Morgan [1998].
tively separable (the borderline case between log-supermodularity and log-submodularity), block segregation obtains (which is only weakly positively assortative setwise).

Our model builds on Smith [2006]. There are two sides of the market, managers and workers, and each manager or worker has a given level of quality. Every manager prefers higher quality workers and every worker prefers higher quality managers. Agents meet in continuous time according to a Poisson process generated by mutual search with quadratic search technology (see Section 2 for definitions). Each pair of agents who have met decide whether to match. A matched agent receives a positive flow of payoffs, which is a function of the agent’s and its matched agent’s qualities (increasing in the quality of the teammate). Unmatched agents receive no payoffs and must continue searching. Agents are impatient, which causes waiting for another match to be costly. All of these features have been used in existing models. We diverge from these models by allowing agents to continue searching for better agents to match with while matched. In the standard model, matched agents exited the market immediately, whereas in our model, they remain on the market, and either one or both sides can continue searching. If a matched agent finds a better match, it can separate the current teammate and rematch with the newly met teammate instead. Separation itself is costless.

Like Smith [2006] and most of the literature discussed above, we operate in a world of nontransferable utility (NTU), where wages are not available to equilibrate matches, and therefore payoffs are not transferable or quasi-linear in a transferable resource (see Shimer and Smith [2000] for the current state of the art in the transferable utility framework). Even though some intramatch transfers (such as making compromises within a match) do exist, the extent of these transfers in many contexts is limited in practice.

As mentioned earlier, we find that block segregation does in fact disappear when we introduce on-the-match search and separation. In particular, if at least one side of the market has the ability to search while matched and to break an inferior match in favor of a better one, then the “banding” of block segregation is no longer a steady-state equilibrium, and strict setwise PAM obtains with weakly log-supermodular payoffs (and thus also with multiplicatively separable payoffs). Furthermore, when both sides are allowed to separate, rather weak conditions on payoffs and search technology guarantee that matching converges to perfect (not just setwise) PAM.

Our paper proceeds as follows. Section 2 outlines the common framework for our analysis and defines various matching patterns that we will be identifying in the various submodels. Section 3 specializes the framework for the baseline case with no on-the-match search and summarizes the results for this case. Section 4 analyzes the case when both sides can search for upgrades while matched and either side can separate from the current match when a better match is found. We

---

5There are no distinguishable sides in Smith [2006]; however, his model can easily be adapted to introduce such a distinction. Furthermore, the two sided-model can be reduced to a model with no sides under symmetry.
will label this *symmetric separation*.\(^6\) Section 5 discusses *asymmetric separation* (only one side of the market can search to upgrade).\(^7\) Section 6 concludes.

2 Matching with Search

The model we present encompasses three distinct submodels: no separation, symmetric separation, and asymmetric separation. However, all three submodels share the same basic setup, the only difference being in upgrade search possibilities. Therefore we use this section to set up a common basic framework for all three models. This approach will necessarily make some definitions somewhat vague (to be made precise only in the sections dealing with the particular sub-models). It will, however, help avoid the redundancy that would result from defining each model from scratch.

The market has two distinguishable sides, or *supertypes*, of agents: managers (M) and workers (W). Both supertypes have equal mass, which we normalize to one. Agents are heterogeneous in terms of their quality; each agent’s type (quality) is distributed on the interval \([0, 1]\). We will use the terms “quality” and “type” interchangeably. Both supertypes of agents have the same type distribution, which is atomless and given by the p.d.f. \(l\). This p.d.f. is everywhere positive and boundedly finite: \(0 < l < l(x) < \bar{l} < \infty\) for any \(x \in [0, 1]\). For brevity, we will use the term “\(x\)-manager” to refer to a manager of quality \(x\), and similarly “\(y\)-worker” for worker of quality \(y\).

Time is continuous and infinite. We will focus on the steady state of the model, where strategies, type distributions, and value functions are time-invariant. Agents meet potential matches randomly according to a Poisson process. The Poisson rate of the meeting process is determined according to quadratic search technology [Smith, 2002]; that is, the rate of meeting an agent in a given subset is proportional to the measure of agents in that subset. The coefficient of proportionality, dubbed the “rendezvous rate” by Smith [2002], is determined by the search intensities of agents, which are determined exogenously. We will assume that the search intensity of all agents who are allowed to search is the same; we call the resulting rendezvous rate \(\rho\).\(^8\)

An unmatched agent receives the flow payoff of zero. An agent of quality \(x\) who is matched to an agent of quality \(y\) receives flow payoff \(f(x, y) > 0\), with \(f\) a continuous function, such that partial

\(^6\)Note that the key feature here is really the ability to search while matched—the ability to separate without on-the-match search would make no difference in a steady-state model, since a match that was once acceptable in a static environment remains acceptable forever. Nonetheless, we use the word “separation” instead of “on-the-match search and separation” for brevity.

\(^7\)The asymmetric model is also equivalent to one where managers can separate only after obtaining consent from their worker. This follows because in the steady state such consent will never be given.

\(^8\)It can be shown (details available upon request) that the results continue to hold when matched agents’ search intensities are lower than unmatched agents’, provided that the difference is sufficiently small.
derivatives with respect to both arguments and the cross-partial derivative exist everywhere.\(^9\) (Note that this also implies that \(f\) is bounded above by \(\bar{f} < \infty\) on \([0, 1]^2\).) The payoff is strictly increasing in the matching agent’s quality (i.e., \(f_2(x, y) > 0\) everywhere). Payoffs are discounted at interest rate \(r > 0\). The present value to \(x\) of being matched to \(y\) forever is thus \(\int_0^{\infty} e^{-rt} f(x, y) \, dt = f(x, y)/r\).

We will be particularly interested in payoff functions that exhibit complementarities between the agents’ qualities (though we will not be assuming the existence of complementarities from the outset). The simplest way to capture the idea of complementarity is that of simple supermodularity.

**Definition 1.** Let \(S \subset \mathbb{R}^2\) and let the cross-partial derivative of \(\phi : S \to \mathbb{R}\) exist. Then \(\phi(x, y)\) is **supermodular** if \(\phi_{12}(x, y) > 0\) for all \(x\) and \(y\). \(\phi\) is **strictly supermodular** if the inequality is strict everywhere.

While the above definition neatly captures the everyday understanding of complementarity, in some cases a stronger form of the concept, log-supermodularity, will be necessary:

**Definition 2.** Let \(S \subset \mathbb{R}^2\) and let the first partial derivative of \(\phi : S \to \mathbb{R}_+\) with respect to the first argument exist. Then \(\phi(x, y)\) is **log-supermodular** if \(\phi_1(x, y_2)/\phi(x, y_2) \geq \phi_1(x, y_1)/\phi(x, y_1)\) for all \(x\) and \(y_2 > y_1\), \(\phi\) is **strictly log-supermodular** if the inequality is strict everywhere.

Observe that the weak inequality holds with equality everywhere if and only if \(\phi\) is multiplicatively separable. This is the standard special case of log-supermodularity.

In addition to the possibility of voluntary separation, a small fraction of matches is randomly dissolved by nature. Dissolution of any given match follows a Poisson process with rate \(\delta\). The random dissolution of matches is necessary to ensure a steady supply of unmatched agents. Random dissolution of matches was introduced as an instrument for this purpose by Shimer and Smith [2000] and Smith [2006]. Other approaches include a steady inflow of new agents [Burdett and Coles, 1997] and replacement of newly matched agents by clones [for example, Bloch and Ryder, 2000; McNamara and Collins, 1990; Morgan, 1998]. Both the cloning and inflow approaches, although fine for the no-separation case, are awkward in our separation setup because matched agents in our model do not exit the market. The population would not remain constant unless inflows were balanced by outflows (deaths), whose rates would have to be matched to inflow, match, and separation rates.

\(^9\)Note that this setup imposes symmetry: an \(x\)-manager who is matched to a \(y\)-worker gets the same payoff as an \(x\)-worker who is matched to a \(y\)-manager. Therefore, an \(x\)-manager who matches a \(y\)-worker receives a payoff \(f(x, y)\), while a \(y\)-worker who matches an \(x\)-manager receives a payoff of \(f(y, x)\). The total surplus is the sum of these two payoffs: \(f(x, y) + f(y, x)\). Because utility is nontransferable, we focus on individual flow payoffs rather than a division of the joint surplus. The TU matching literature focuses more on division of surplus and transfers between the two sides of the market.
Strategies and Value Functions

A worker’s or manager’s strategy consists of:

1. An agent’s acceptance set, i.e., the set of matching agents to which this agent is willing to match to when unmatched. Let \( A^i(x) \) be the acceptance set of an unmatched worker of supertype \( i \in \{W, M\} \) and type \( x \). Let \( \alpha^i(x, y) \) be the indicator function for \( y \in A^i(x) \). Thus, for example, \( \alpha^W(x, y) = 1 \) if and only if an \( x \)-worker is willing to match a \( y \)-manager, and \( \alpha^W(x, y) = 0 \) otherwise.

2. A matched agent’s acceptance set, i.e., the set of partners to which this agent is willing to upgrade when already matched. Let \( A^i(x \mid y) \) be the acceptance set of an worker of supertype \( i \in \{M, W\} \) and type \( x \) when matched to manager of type \( y \). Let \( \alpha^i(x, z \mid y) \) be the indicator function for \( z \in A^i(x \mid y) \). For example, \( \alpha^M(x, z \mid y) = 1 \) if and only if an \( x \)-worker who is currently matched to a \( y \)-manager is willing to unmatched from the manager and match a \( z \)-manager instead, and \( \alpha^M(x, z \mid y) = 0 \) otherwise.

The expected average present value to an \( x \)-worker of supertype \( i \) who is unmatched is \( V^i(x) \). The expected average present value to an \( x \)-worker who is matched to \( y \) is \( V^i(x \mid y) \). Note that these are average values. The actual expected present values are therefore \( V^i(x)/r \) and \( V^i(x \mid y)/r \). However, for brevity, we will use the term “value function” throughout the paper to refer to the expected average present value.

The density of workers of type \( x \) who are matched to managers of type \( y \) is given by \( \mu(x, y) \). It will also be convenient to denote \( \mu^M(x, y) \equiv \mu(x, y) \) and \( \mu^W(x, y) \equiv \mu(y, x) \). Note that the mass of matched agents of a given type cannot exceed the total mass of agents of that type, so that a given function \( \mu : [0, 1]^2 \rightarrow \mathbb{R}_+ \) is an admissible match density function if and only if \( \int_0^1 \mu(x, y) \, dy \leq l(x) \) for all \( x \) and \( \int_0^1 \mu(x, y) \, dx \leq l(y) \) for all \( y \).

The match density function defines also the densities of unmatched agents of supertype \( i \) and type \( x \), which we denote \( \nu^i(x) \). Note that \( u^i(x) = l(x) - \int_0^1 \mu^i(x, y) \, dy \) and that \( \int_0^1 u^M(x) \, dx = \int_0^1 u^W(x) \, dx = 1 - \int_0^1 \mu(x, y) \, dx \, dy \). Finally, given a supertype \( i \) and a type \( x \), the matching set of \( x \)-workers of supertype \( i \) will be defined as the set of managers of supertype \( -i \) that this worker can be matched to in equilibrium:

\[
\mathcal{M}^i(x) = \{ y \mid \mu^i(x, y) > 0 \}.
\]

---

\(^{10}\)Suppose worker \( x \) has expected present value \( \nu^i(x) \). Then the average present value \( V^i(x) \) is the constant flow payoff that this worker would have to receive from now to infinity in order to get the same average present value \( \nu^i(x) \): \( \nu^i(x) = \int_0^\infty e^{-rt} V^i(x) \, dt = V^i(x)/r \).
Steady state

A steady state is defined as a situation where all relevant elements of the model are time-invariant. In fact, this is equivalent to saying that match densities are time invariant, since stationary strategies and stationary value functions arise naturally in a stationary environment. Thus, a steady state is given by the condition that the relevant match creation rate is everywhere equal to the match separation rate.

More precisely, in models with separation, the steady state will be defined by the condition that the rate at which managers of type $x$ get matched to workers of type $y$ is equal to the total dissolution rate of $(x, y)$ matches (for each $x$ and $y$). In the model without separation, the steady state will be defined by the condition that the rate at which agents of type $x$ get matched equals the rate at which unmatched agents of type $x$ enter the model (through random match dissolution). Note that the definition of steady state in the no-separation model refers only to unmatched densities, instead of match densities. This is because the exact match density is irrelevant for agents’ strategies in that case, since an agent’s opportunity set does not depend on who is matched to whom: a matched agent is an unavailable agent, no matter to whom it is matched.

Search equilibrium

A search equilibrium of the model is given by an admissible match density function $\mu$, acceptance sets $A^i$, and value functions $V^i$ satisfying the following conditions:

1. Steady state;

2. Value functions consistent with the expected payoffs actually obtained, as given by value function equations;

3. Rational strategies, i.e., $\alpha^i(x, y) = 1$ if and only if $V^i(x | y) \geq V^i(x)$ and $\alpha^i(x, z | y) = 1$ if and only if $V^i(x | z) > V^i(x | y)$.

2.1 Types of Matching Patterns

The key question in any steady-state matching model is who matches with whom. In particular, when do higher quality agents match to higher quality partners? The most straightforward version of like-to-like matching is perfect positively assortative matching, whereby each agent matches to its own type:

**Definition 3.** There is perfect positively assortative matching (perfect PAM) if and only if $M^i(x) = \{x\}$ for all $x \in [0, 1]$ and $i \in M, W$. 
This concept is appealing not only because of its simplicity but also because perfect PAM is the unique matching pattern that maximizes total surplus (see Becker [1973]) when payoffs are strictly supermodular.

However, perfect PAM generally cannot be achieved in our model because the search cost imposed by time discounting results in non-singleton acceptance and matching sets. This forces us to look for weaker definitions of assortative matching: i.e., ones where higher quality agents match to higher sets of agents. The appropriate standard concept, due to Shimer and Smith [2000] and Smith [2006], is setwise PAM:

**Definition 4.** There is setwise positively assortative matching (setwise PAM) if, for each \( i \in \{M, W\} \), \( x_1 < x_2 \), and \( y_1 < y_2 \) such that \( y_1 \in \mathcal{M}^i(x_2) \) and \( y_2 \in \mathcal{M}^i(x_1) \), it is also true that \( y_1 \in \mathcal{M}^i(x_1) \) and \( y_2 \in \mathcal{M}^i(x_2) \). There is strict setwise PAM if, for each \( x_1 < x_2 \) and \( y_1 < y_2 \) such that \( y_1 \in \mathcal{M}^i(x_2) \) and \( y_2 \in \mathcal{M}^i(x_1) \), it is also true that \( y_1 \in \text{int} \mathcal{M}^i(x_1) \) and \( y_2 \in \text{int} \mathcal{M}^i(x_2) \).

As Shimer and Smith [2000] show, when managers’ matching sets are nonempty, setwise PAM obtains if and only if matching sets are intervals with weakly increasing upper and lower bounds. Strict setwise PAM obtains if and only if the bounds are strictly increasing, except possibly when they are equal to zero or one. Finally, note that, since agents’ matching correspondences are inverses of each other, it is sufficient to check the conditions of the definition for just one of \( i \in \{M, W\} \). This clearly shows how setwise PAM corresponds to the idea of higher-quality agents matching with higher-quality sets of partners. However, it should also be noted that setwise PAM is a much weaker condition than perfect PAM. For example, a situation where \( \mathcal{M}^W(x) = [0.001x, 1] \) for all \( x \) constitutes strict setwise PAM, even though all agents’ matching sets are virtually identical. In fact, \( \mathcal{M}^W(x) = [0, 1] \) also implies strict setwise PAM, despite the fact that matching sets are identical, and there is no actual sorting.

Block segregation is a particular form of weak setwise PAM, whereby agents separate themselves into disjoint matching classes, such that no matching occurs across classes:

**Definition 5.** There is block segregation (banding) if the type space partitions into at least two (and possibly infinitely many) disjoint classes \([\theta_1, \theta_0] \cup [\theta_2, \theta_1] \cup \ldots\) with \( 1 = \theta_0 > \theta_1 > \theta_2 > \ldots \) such that, for each \( i \in \{M, W\} \), \( x \in \mathcal{M}^i(y) \) if and only if \( x \) and \( y \) are in the same class.

Note that the upper and lower bounds of matching sets are constant almost everywhere and discontinuously increasing at a finite or countably infinite set of points. In particular, this matching pattern is not strict setwise PAM.

---

11In addition, note that a perfect PAM distribution is degenerate. Hence, the density \( \mu \) does not exist (the “density” of the distribution is a Dirac delta function). Thus perfect PAM is not consistent with search equilibrium as defined above.
The definitions of the matching patterns described above are now standard in the matching literature. Initially defined in the context of the no-separation model, they are based on matching sets alone. In the context of a model with search and upgrade, however, the matching sets do not come close to telling the full story: because of the possibility of separation, different matches within the matching set have different expected durations and different equilibrium densities. In addition to asking which matches are possible in principle, we now must also ask which matches are likely to last. We therefore need a new characterization of assortative matching in the context of our modified models. Notice also that when the exogenous match dissolution rate is very high, few agents are able to upgrade from one match to another before they find themselves available again. In this situation, matching patterns are determined mostly by initial matching decisions, and there are only small efficiency gains from the possibility of on-the-match search. Since we are interested in the matching patterns arising from agents’ ability to upgrade matches, which creates a dynamic sorting mechanism, we want to focus on situations when the exogenous dissolution rate is negligible. This leads us to the following definition:

**Definition 6.** The matching pattern *converges in* $\delta$ *to perfect PAM* if for all $R > 0$

$$
\lim_{\delta \to 0^+} \sup_{x \in [0,1]} \left| l(x) - \int_{x-R}^{x+R} \mu(x, y) \, dy \right| = 0.
$$

The definition says that the model converges in $\delta$ to perfect PAM if the mass of matches becomes increasingly concentrated around the diagonal ($\{(x, x) \mid x \in [0, 1]\}$) as the external match dissolution rate goes to zero. In other words, the matching pattern converges in $\delta$ to perfect PAM if the equilibrium matching measure weakly converges to the perfect PAM measure.

## 3 Baseline Case: No Separation

We are now ready to specify and solve each of the three sub-models: no separation, symmetric separation, and asymmetric separation. We begin with the baseline case of no separation. This is the model that has been widely studied in the literature. This section includes no new results: all the findings in this section are due to Smith [2006]. We state the results for this case as a benchmark for evaluating the effects of adding the possibility of separation.

### 3.1 Model Specification

In this case, only unmatched agents can search, and matched agents’ acceptance sets $A^i(x \mid y)$ are constrained to be empty. Furthermore, the exact matching density is strategy-irrelevant. Therefore, in the definition of equilibrium, we replace the match density function $\mu$ with the unmatched densities $u^i$. Smith [2006] proves that there is a unique equilibrium distribution of unmatched
agents. (A continuum of matching densities is compatible with this equilibrium, but they all have the same matching sets, because $\mu(x, y) > 0$ if and only if $\alpha^M(x, y) = \alpha^W(y, x) = 1$.) The value functions are given by

$$V^i(x) = \frac{\rho}{r} \int_0^1 \alpha^i(x, y) (V^i(x | y) - V^i(x)) u^{-i}(y) \alpha^{-i}(y, x) \, dy$$

$$= \frac{\rho}{r} \int_0^1 \max (V^i(x | y) - V^i(x), 0) u^{-i}(y) \alpha^{-i}(y, x) \, dy. \quad (1)$$

and

$$V^i(x | y) = f(x, y) + \frac{\delta}{r} (V^i(x) - V^i(x | y)). \quad (2)$$

The steady state equation is

$$\delta \left( l(x) - \int_0^1 u^i(x) \, dx \right) = u^i(x) \rho \int_0^1 u^{-i}(y) \alpha^i(x, y) \alpha^{-i}(y, x) \, dy. \quad (3)$$

When symmetry of the strategies of the two supertypes is imposed, this no-separation model reduces to that of Smith [2006], and thus the results in that paper hold. Furthermore, Smith also notes that the results extend to the non-symmetric case. We therefore turn to the description of the matching patterns.

### 3.2 Matching Patterns

The first key observation in Smith’s model is that block segregation (see Definition 5) obtains whenever payoffs are multiplicatively separable in the agent’s qualities. This is a common observation in the literature (see Section 1), though Smith [2006] was the first to show it with general multiplicatively separable payoffs (other papers used particular functional forms, such as $f(x, y) = y$ or $f(x, y) = xy$). The logic of Smith’s proof consists of two steps. First, he observes that multiplicative payoffs yield identical von Neumann-Morgenstern preferences over matches. Second, search frictions create a highest band of agents, who are accepted by everybody, because their quality exceeds the time cost of waiting for another meeting, even for an agent who can be sure that nobody will reject her. Thus agents in the highest band have not only the same preferences, but also the same opportunities. Consequently, they must make the same choices. Proceeding recursively to ever lower bands yields the overall block segregation result.

**Proposition 1.** Assume $f(x, y) = \varphi_1(x)\varphi_2(y)$ for functions $\varphi_1, \varphi_2$, with $\varphi_1 > 0$. Then there is block segregation in the no-separation model. If $\varphi_2(0) = 0$, there are infinitely many segregation classes.

**Proof.** See Proposition 2 and Lemma 7 in Smith [2006].
Note that block segregation relies heavily on the presence of search frictions and the permanence of matches. Waiting is costly, so even the highest-type agent will accept some lower-quality agents. In addition, since a match is permanent, initial matching decisions fully determine lifetime utility, so all agents who are accepted by the highest-type agent (including the highest type agent herself) must have the same opportunity set. Since multiplicative payoffs imply identical cardinal preferences over matches, these agents, who face the same opportunities, make the same decisions.

Intuitively, this outcome can be avoided if the marginal payoffs to a better teammate increase sharply with an agent’s type, so that higher-quality agents become relatively more patient in waiting for better matching agents. In this case, the cardinal preferences over matches are no longer the same for all agents (they increase faster for higher-quality agents), which causes the banding result to break down. Instead, strict setwise PAM holds: agents of strictly higher quality match to sets of teammates of strictly higher quality (see Definition 4). This is the central result of Smith’s paper:

**Proposition 2.** If the payoff function is strictly log-supermodular, there is strict setwise positively assortative matching.

**Proof.** See Smith [2006], Proposition 3.

The proof proceeds by mathematical manipulation of the value function, but its intuition follows the path outlined above. Under the strong form of complementarity exhibited by log-supermodular payoffs (see Definition 2), agents of strictly higher quality find waiting for better matches strictly more desirable, so their acceptance thresholds are strictly higher. While search frictions still imply that matching sets are intervals instead of points, these intervals are now strictly increasing in an agent’s type, and the constant bands from above unravel.

### 4 Symmetric Separation

The previous section shows that matching patterns in a world with no on-the-match search can be quite unsatisfying. First, with multiplicatively separable functions, we obtain an unintuitively discontinuous matching correspondence (banding). Second, even when no banding occurs, the steady-state equilibrium pattern does not typically achieve efficiency. In particular, inefficient matching is observed when payoffs are supermodular, so that the unique total-surplus maximizing matching pattern is perfect PAM, which is never reached in equilibrium. Three levels of inefficiency can be observed in this case: if payoffs are log-supermodular, the equilibrium is strict setwise PAM; when they are multiplicatively separable, the equilibrium is weak setwise PAM (block segregation); but, when payoffs are supermodular, but not log-supermodular, even setwise PAM does not obtain (in fact, negative setwise assortative matching could occur). We therefore introduce the possibility
for agents to continue to search while already matched to an agent and ask ourselves whether this possibility leads to more efficient outcomes.

4.1 Model Specification

Now, all agents are allowed to search while matched and to separate from their current matches when a more desirable match is found. Before writing down the value functions and the steady-state equation, it will be convenient to define a few auxiliary functions. First, for each $i \in \{M, W\}$, let $\Omega^i(x, y)$ be the rate at which an $x$-agent of supertype $i$ meets $y$-agents that are willing to match with it (the opportunity rate):

$$\Omega^i(x, y) = \rho \left( u^{-i}(y) \alpha^{-i}(y, x) + \int_0^1 \alpha^{-i}(y, x | x') \mu^{-i}(y, x') dx' \right). \quad (4)$$

The first summand corresponds to $x$ meeting a unmatched $y$, whereas the second stands for $x$ meeting an already-matched $y$ who is willing to separate and match $x$ instead. Similarly, let $D^i(x, y)$ be the rate at which an $x$-agent of supertype $i$ separates from its match, given that the current teammate’s quality is $y$ (the separation rate):

$$D^i(x, y) = \rho \int_0^1 \alpha^i(x, y' | y) \left[ u^{-i}(y') \alpha^{-i}(y', x) + \int_0^1 \alpha^{-i}(y', x | x') \mu^{-i}(y', x') dx' \right] dy'. \quad (5)$$

The first term in the brackets represents $x$ meeting a unmatched $y'$, whereas the second corresponds to $x$ meeting a $y'$ who is already matched and willing to separate and match $x$ instead.

We can now write down the value functions and the steady-state equation. The value function of an unmatched agent of supertype $i \in \{M, W\}$ is

$$V^i(x) = \frac{1}{r} \int_0^1 \alpha^i(x, y) \left( V^i(x | y) - V^i(x) \right) \Omega^i(x, y) dy$$

$$= \frac{1}{r} \int_0^1 \max \left( V^i(x | y) - V^i(x), 0 \right) \Omega^i(x, y) dy. \quad (6)$$

Since the flow payoff from not being matched is zero, the value comes only from expected future matches. In particular, the expected value is the integral over all possible manager quality levels $y$ of the product of three terms: an indicator whether an agent of type $x$ is willing to match a $y$-agent at all, the value gain from being matched to a $y$-agent, and the rate at which $x$ will meet available teammates of this type. The second line of the equation above follows from the fact that $x$-agents choose $\alpha^i(x, y)$ rationally.
Similarly, the value of a matched agent is

\[ V^i(x \mid y) = f(x, y) + \frac{1}{r} (V^i(x) - V^i(x \mid y)) (\delta + D^{-i}(y, x)) + \frac{1}{r} \int_0^1 \max \left( V^i(x \mid y') - V^i(x \mid y), 0 \right) \Omega^i(x, y') \, dy'. \]  

(7)

The first term represents the payoff from being matched to the current agent; the second term represents the value loss when the match is dissolved by Nature or due to separation by the current agent; the final term stands for the possibility of upgrade to a more desirable agent.

The steady-state equation is

\[ \mu(x, y) [\delta + D^M(x, y) + D^W(y, x)] = u^M(x) \alpha^M(x, y) \Omega^M(x, y) + \int_0^1 \mu(x, y') \alpha^M(x, y \mid y') \Omega^M(x, y) \, dy'. \]  

(8)

The left-hand side is the match dissolution rate at \((x, y)\), while the right-hand side is the match formation rate at \((x, y)\). The first term on the right represents new \((x, y)\)-matches involving a unmatched \(x\)-worker, while the second term stands for new matches involving an \(x\)-worker who was matched to someone else when it met its \(y\)-manager (note that this side of the equation could equivalently be expressed by integrating over workers rather than managers).

Finally, we also require that

\[ \alpha^i(x, y) = 0 \Rightarrow \mu^i(x, y) = 0. \]  

(9)

This is because \(\alpha^i(x, y) = 0\) implies \(V^i(x \mid y) < V^i(x)\) (by rationality of strategies), and, since all agents are free to separate, all such matches would instantly dissolve.

4.2 Equilibrium and Convergence

We will use a constructive approach to equilibrium analysis: we will explicitly construct an equilibrium, which will simultaneously prove that an equilibrium exists and characterize that equilibrium. We begin by assuming that a search equilibrium exists and establish some basic facts that must be true in any equilibrium. We then conjecture that a particular strategy profile gives rise to a search equilibrium. We prove that this is indeed the case by explicitly constructing the unique match densities and value functions that arise from the conjectured equilibrium strategies and then showing that the constructed value functions do in fact imply that these strategies are optimal. Finally, we use the construction from the existence proof to show that the resulting equilibrium matching pattern converges in \(\delta\) to perfect PAM.

Suppose a search equilibrium exists. The first observation we make is that every unmatched agent initially accepts any possible matching agent it meets. This should not be surprising; because
matched agents can continue searching for upgrades, unmatched agents do not give anything up by accepting a match, and they gain some immediate payoff from being matched. Thus, being matched to anyone is strictly preferable to remaining unmatched.

**Lemma 1.** Everyone always accepts everyone when unmatched: \( \alpha^i(x, y) = 1 \) for all \( x, y \in [0, 1] \) and \( i \in \{M, W\} \). Furthermore, the preference for matching over remaining unmatched is strict for all agents.

**Proof.** See Appendix.

Another useful observation is that \( V^i(x \mid y) - V^i(x) \) is everywhere less than \( f(x, y) \). The intuition for this result is clear: \( f(x, y) \) is the value to an \( x \)-worker of being in a perpetual match with a \( y \)-manager. Thus, \( f(x, y) \) would be the difference between the value of having the right to stay matched to a \( y \)-manager forever and the value of being unmatched. Since an actual match with a \( y \)-manager is less valuable than the right to stay matched with that manager forever (because the other party in the match can initiate a separation), the difference of the value of an actual match with a \( y \)-manager and the value of being unmatched is less than \( f(x, y) \).

**Lemma 2.** \( V^i(x \mid y) - V^i(x) < f(x, y) \) for all \( x, y \in [0, 1] \) and \( i \in \{M, W\} \).

**Proof.** See Appendix.

Next, observe that the direction of change in \( V^i(x \mid y) \) in response to changes in \( y \) is determined entirely by changes in the terms corresponding to current payoff and the possibility of separation, i.e., by changes in the expression \( f(x, y) + \frac{1}{r}(V^i(x) - V^i(x \mid y))(\delta + D_{i}^{-i}(y, x)) \) in (7). This is because the upgrade possibilities do not depend on the quality of the agent to which one is current matched; since \( V^i(x) - V^i(x \mid y) < 0 \), the expression above represents a trade-off between higher current payoffs and higher possibility of separation initiated by the teammate. It also follows that where the partial derivatives \( V^i_2(x \mid y) \) and \( D_{i}^{-i}(y, x) \) exist, the sign of \( V^i_2(x \mid y) \) is determined by \( f_2(x, y) \) and \( D_{i}^{-i}(y, x) \). More precisely,

**Lemma 3.** Wherever \( V^i(x \mid y) \) and \( D_{i}^{-i}(y, x) \) are differentiable with respect to \( y \), the sign of \( V^i_2(x \mid y) \) is determined by the following identity:

\[
\text{sgn} \left( V^i_2(x \mid y) \right) = \text{sgn} \left( f_2(x, y) + \frac{1}{r}(V^i(x) - V^i(x \mid y))D_{i}^{-i}(y, x) \right).
\]

**Proof.** See Appendix.

The result of Lemma 3 does not immediately imply monotonicity of \( V^i(x \mid y) \) and is thus in principle compatible with many strategies. However, we consider it natural to look for an equilibrium in simple, symmetric, and intuitive strategies. In particular, we conjecture that there is
an equilibrium in which all matched workers or managers upgrade when possible. Consider the following conjecture:

**Conjecture 1.** There is a search equilibrium in which $\alpha^M(x, y) = \alpha^W(x, y) = 1$ for all $x$ and $y$, and, for any $i \in \{M, W\}$, $\alpha^i(x, z \mid y) = 1$ if and only if $z > y$.

This conjectured strategy ("accept when unmatched, upgrade when possible") gives rise to a well-defined, symmetric match density $\mu$ and continuous value functions $V(x)$ and $V(x \mid y)$, which are the same for managers and for workers. In addition, $V(x \mid y)$ is differentiable with respect to $y$ for any $x$ (these results are proven in the Appendix). To establish an equilibrium, the only remaining step is to show that it is optimal for every agent to follow this strategy, given these match densities and value functions and given that all other agents are following this strategy. Lemma 1 already showed that it is optimal for every agent to accept any other agent when unmatched. Thus, we only need to find sufficient conditions under which it is strictly optimal for every agent to always upgrade when possible. That is, we need to find conditions under which the $V(x \mid y)$ derived above is strictly increasing in $y$.

If everybody uses the conjectured strategy, the separation rate is given by

$$D(x, y) = \rho \int_y^1 \left[ u(y') + \int_x^\infty \mu(y', x') \, dx' \right] \, dy',$$

which is increasing in the first argument and differentiable with respect to that argument, with

$$D_1(x, y) = \rho \int_y^1 \mu(y', x) \, dy'.$$  \hspace{1cm} (10)

To interpret this, suppose that the manager of quality $x$ in the expression is a manager (who is matched to a $y$-worker). Call him Manager X. Then the expression equals the rate at which Manager X meets workers who are better than his current worker and are currently matched to $x$-managers. Suppose Manager X’s quality increases slightly. Then the expression for $D_1(x, y)$ is precisely the additional pool of desirable workers to whom the $x$-manager can now upgrade. Before the quality increase, the $y$-workers who were matched to other $x$-managers would not have to dissolve their match with their managers for Manager X, but they are happy to do so after the slight increase in his quality.

Since $V(x \mid y)$ is differentiable with respect to its second argument and $D(x, y)$ is differentiable with respect to its first argument, Lemma 3 applies. That is, $V(x \mid y)$ is everywhere increasing in $y$ if and only if

$$f_2(x, y) + \frac{1}{r}(V(x) - V(x \mid y))D_1(y, x) > 0.$$  \hspace{1cm} (11)

This condition holds if and only if the payoffs $f(x, y)$ are increasing in $y$ sufficiently quickly, relative to the increase in the separation rate due to $y$. When everybody else is using the conjectured
strategy, $D_1(y, x)$ is proportional to the density of $y$-workers whose matched managers are better than $x$ (by (10)), which is in turn bounded above by the total density of $y$-workers, $l(y)$. Noting that the difference $V(x|y) - V(x)$ is less than $f(x, y)$ (by Lemma 2), we can now replace the condition (11) by a simple condition on the primitives of the model: the relative increase in the payoffs at $(x, y)$ due to an increase in $y$ must be at least as large as the density of agents at $y$ times $\rho/r$, for any $x$ and $y$.

**Lemma 4.** Let

$$\frac{f_2(x, y)}{f(x, y)} \geq \frac{\rho}{r} l(y) \quad \text{for all } x \text{ and } y.$$  

Then the strategy in Conjecture 1 is the unique best response to itself.

**Proof.** Let the conditions in the statement of the lemma hold, and let everybody (except a given $x$-worker) use the conjectured strategy. If the worker is unmatched, the conjectured strategy is uniquely optimal by Lemma 1. If the worker is matched to some $y$-manager, Lemma 3 implies, as explained above, that the conjectured strategy is the unique optimal response if and only if the condition (11) holds. By (10) and the admissibility of $\mu$, $D_1(y, x) \leq \rho l(y)$. By Lemmas 2 and 1, $0 < V(x|y) - V(x) < f(x, y)$. Consequently, the second summand in condition (11) is greater than $-(\rho/r)f(x, y)l(y)$, which by the assumption of Lemma 4 is no less than $-f_2(x, y)$. But then (11) holds.

We now arrive at our first main result.

**Theorem 1.** If

$$\frac{f_2(x, y)}{f(x, y)} \geq \frac{\rho}{r} l(y) \quad \text{for all } x \text{ and } y,$$

then there exists a search equilibrium in which all agents follow the strategy given by $\alpha(x, y) = 1$ for all $x$ and $y$; $\alpha^t(x, z|y) = 1$ if and only if $z > x$. Furthermore, the corresponding equilibrium match density is symmetric, $\mu(x, y) = \mu(y, x)$ for all $x$ and $y$.

**Proof.** See Appendix.

Note that the condition in the hypothesis always holds for a payoff function with sufficiently high relative marginal benefit from being matched to a better teammate ($f_2(x, y)/f(x, y)$): when there are large gains from upgrades, these upgrades will take place. Second, for any payoff function, the condition holds for sufficiently high interest rate $r$. As individuals become more impatient, the immediate gain from an upgrade outweighs the possible future loss from an increased risk of separation. In fact, this theorem is stronger than the claim that “accept everyone initially and upgrade whenever possible” is part of a search equilibrium. We prove that this strategy is the
unique best response to itself in the stationary environment that it generates. That is, it is an
evolutionarily stable strategy (ESS), as defined by Smith and Price [1973].

The proof of Theorem 1 takes place in several steps. The first task is to show that the conjectured
strategy “accept when unmatched, upgrade when possible” induces a well defined steady state
match distribution. This involves defining a best response function that has a unique fixed point.
While ordinarily this approach is straightforward, the operator defining the steady-state matching
density is not a contraction, which requires a more complicated solution technique. In particular, we
discretize the problem and solve recursively, and then show that the unique solutions of a sequence
of discretized problems converge to the unique solution of the original continuous problem. This
establishes that the conjectured equilibrium strategy gives rise to a well defined, symmetric match
density function. The final step simply involves solving for the value functions. It follows that the
search equilibrium exists and the corresponding equilibrium match densities are symmetric.

Now that we have established the existence of an equilibrium and derived an algorithm for
deriving its match density, we can turn to investigating the properties of the emerging matching
pattern. We will use the discretized model developed in the proof of the main theorem above to
help us establish the equilibrium matching pattern under the equilibrium found in the previous
section.

Consider the highest-type agents in the discrete model with $\delta = 0$. Because there is not external
dissolution and no worker leaves a highest-type manager and no manager fires a highest-type worker,
we know that no highest-highest matches are ever dissolved, once they are formed. Thus, there is
no outflow from this type of match. By the definition of steady state, there must therefore be no
inflow to such matches either. But this implies that there cannot be any highest-type managers
with out a job or employing a lower-type worker or highest-type workers without a manager or
working for a lower-type manager, because whenever such agents meet, they will match and create
inflow into highest-highest matches. Thus, all highest-type agents are matched to highest-type
agents. We can repeat this argument recursively, proceeding to ever lower types, to conclude that
all agents are matched to agents of their own type.

Note that $\delta = 0$ does not yield any search equilibrium (as defined above) in the original contin-
uous model. The existence proof in the previous section breaks down. The matching “density” is a
Dirac $\delta$ function. Nonetheless, the case with $\delta = 0$ in the discrete case suggests that the matching
pattern in the continuous model should converge in $\delta$ to perfect positively assortative matching
(recall Definition 6). This intuition turns out to be correct:

---

12The idea here is that this strategy maximizes one’s payoff or, using the original biological term, *evolutionary fitness* in a population where everyone plays this strategy, and that no alternative strategies can do equally well. Thus if subpopulations using an alternative strategy were to arise, they would lose out in the *survival of the fittest*.
Theorem 2. When the equilibrium established above exists, the corresponding matching pattern converges in $\delta$ to perfect PAM.

Proof. See Appendix.

The key criteria in this theorem is that $\delta/\rho$ converges to 0 (therefore $\delta$ converging to 0 is sufficient). In words, the external dissolution rate relative to the rendezvous rate must vanish, so the rate at which matches dissolve shrinks relative to the rate of agents meeting other agents. Recall that the external match dissolution rate is necessary for a steady state model to exhibit a search equilibrium. While $\delta = 0$ may seem more intuitively plausible, a positive $\delta$ is necessary in order for new agents to continually enter the market. This is essentially a technical condition on the model. The convergence result, therefore, should be reassuring because it argues that as this technical condition vanishes, the equilibrium converges to the “good” equilibrium of perfect positive assortative matching.

The proof of theorem 2 proceeds by first constructing a discrete equilibrium and then applying convergence results to show that the equilibrium of the continuous model is close to that of the discrete model. When formation of entirely new matches by unmatched agents (either new arrivals or agents left unmatched as the result of match dissolution) is dominated by long-term upgrade dynamics among already-matched individuals (i.e., the dissolution rate $\delta$ is low), the matching pattern corresponding to the equilibrium we have identified approaches perfect positively assortative matching.

This is reassuring for many reasons. First, it suggests that divorce, an important feature of real-life marriage markets, has “good” results for the model. The matching literature has identified positively assortative matching as the gold standard, and for years block segregation has plagued the equilibrium of two sided search and matching models. While Smith (2006) varied the payoff functions to show under what conditions setwise PAM obtains, we keep the payoff functions fixed, but rather add a new institutional feature to the model, namely separation. Allowing agents to search while matched and separate if desired both captures an important feature of reality, and eliminates this pathological equilibrium of block segregation. Thus we see our result as complementary to that of Smith (2006); both papers give conditions under which the two-sided matching model with search can arrive at positive assortative matching.

Figure 3 shows the match density on the path to convergence. The figure plots the contours of the match density function, which is a joint density over the two type spaces, one for each side of the market. The lighter colored regions denote more mass of the density. The four pictures represent the level sets of the match density functions at different parameter choices of $\delta/\rho$. As $\delta/\rho$ shrinks toward zero, observe that the matching density places more mass along the diagonal, where the diagonal represent perfect assortative matching. Also note that the higher types converge faster,
as the density is tighter and more concentrated around the upper region of each type space. As the random dissolution rate vanishes, the match density function converges to the 45 degree line, and perfect positively assortative matching obtains.

Figure 4 shows convergence also as $\delta$ shrinks to zero. Observe that as $\delta$ becomes small, the
Figure 4

- % matched
- % matches with less than 0.2 quality difference
- Total surplus as a percentage of max when $f(xy) - xy$
percentage matched perfectly increases as well as the percentage matched within a 20 percent quality difference. And, if we assume a symmetric payoff function of $f(x, y) = xy$, observe that total surplus as a percentage of maximum surplus also rises as $\delta$ shrinks. In words, as the dissolution rate vanishes, agents are more closely matched to their own type, and this more precise matching increases total surplus.

5 Asymmetric Separation

In this final extension of the model, we’ll investigate whether allowing just one side of the market to search while matched is sufficient to induce stricter assortative matching than in the absence of on-the-match search.

5.1 Model Specification

Let us assume that workers (without loss of generality) can continue searching while matched. If they find a better match, the previous one is dissolved and a new one is formed instead. Managers, on the other side, cannot search while matched and cannot initiate a separation. The value function for a unmatched worker is

$$V_W(x) = \frac{1}{r} \int_0^1 \alpha_W(x, y) \left( V_W(x \mid y) - V_W(x) \right) \rho u^M(y) \alpha^M(y, x) \, dy$$

$$= \frac{\rho}{r} \int_0^1 \max \left( V_W(x \mid y) - V_W(x), 0 \right) u^M(y) \alpha^M(y, x) \, dy. \quad (12)$$

For each acceptable type $y$ of managers, the contribution to the expected payoff is the value to the $x$-worker of being matched to a $y$-manager times the rate at which the $x$-worker can expect to meet available $y$-managers. The value function for a matched worker is

$$V_W(x \mid y) = f(x, y) + \frac{\delta}{r} (V_W(x) - V_W(x \mid y))$$

$$+ \frac{\rho}{r} \int_0^1 \max \left( V_W(x \mid y') - V_W(x \mid y), 0 \right) u^M(y') \alpha^M(y', x) \, dy'. \quad (13)$$

The first term represents the payoff from being matched to the current teammate; the second term represents the payoff when the match is dissolved by Nature; the third stands for the possibility of upgrade to a more desirable match.

Before we write down the value functions for managers and the steady-state equations, it will be useful to introduce two more pieces of notation. First, let us denote the rate at which an $x$-manager hires a $y$-worker by $\Omega(x, y)$ (where $\Omega$ stands for “opportunity rate”):

$$\Omega(x, y) = \rho \left( u^W(y) \alpha^W(y, x) + \int_0^1 \alpha^W(y, x \mid x') \mu^W(y, x') \, dx' \right). \quad (14)$$
The first term in parentheses represents our $x$-manager finding a $y$-worker who would accept him, while the second represents him finding an already employed $y$-worker who is willing to leave his current manager for him. Note that when no $y$-workers are willing to accept $x$, $\Omega(y, x) = 0$. Second, let us denote the dissolution rate of a match between an $x$-manager and a $y$-worker by $D(x, y)$:

$$D(x, y) = \delta + \rho \int_0^1 \alpha_W(y, x') | x)u^M(x')\alpha^M(x', y) dx'. \quad (15)$$

The first term stands for dissolution by Nature, while the second stands for dissolution due to the worker meeting a better match and leaving its current manager.

Now, we can concisely write down the value functions of managers and the steady-state equations. The value function for an available manager is

$$V^M(x) = \frac{1}{r} \int_0^1 \alpha^M(x, y) \left( V^M(x | y) - V^M(x) \right) \Omega(x, y) dy$$

$$= \frac{1}{r} \int_0^1 \max \left( V^M(x | y) - V^M(x), 0 \right) \Omega(x, y) dy. \quad (16)$$

The value function for a matched manager is

$$V^M(x | y) = f(x, y) + \frac{1}{r}(V^M(x) - V^M(x | y))D(x, y). \quad (17)$$

The steady state equation is (for each $x$ and $y$)

$$\mu(x, y)D(x, y) = \alpha^M(x, y)u^M(x)\Omega(x, y). \quad (18)$$

The left-hand side is the match dissolution rate at $(x, y)$, while the right-hand side is the rate at which new $(x, y)$-matches are formed. Note that the right-hand side is zero when an $x$-manager does not accept a $y$-worker. If the manager accepts, the formation rate is proportional to the mass of unmatched managers (since only unmatched managers are allowed to form new matches) and the rate at which a given unmatched manager meets $y$-workers who accept him.

In addition to the steady-state equation above, we also require that

$$\alpha^W(y, x) = 0 \Rightarrow \mu(x, y) = 0. \quad (19)$$

This is because $\alpha^W(y, x) = 0$ implies $V^W(x | y) < V^W(x)$ (by rationality of strategies), and, since workers are free to separate, all such matches would instantly dissolve.

5.2 Strategies

The first observation is that managers act as if separation were not a possibility. That is, in deciding whether to accept a match with a worker, they simply compare the flow value of being matched to
that worker forever to the value of being unmatched. This is immediate from the value-function equation for matched managers: reordering the terms in (17), we get

\[ V^M(x|y) - V^M(x) = \frac{r}{r + D(x,y)}(f(x,y) - V^M(x)). \]  

(20)

Since \( r \) and \( D(x,y) \) are positive, it follows that the sign of \( V^M(x|y) - V^M(x) \) equals the sign of \( f(x,y) - V^M(x) \). Since \( f(x,y) \) is strictly increasing in \( y \), we know that if \( f(x,y_0) - V^M(x) > 0 \), then \( f(x,y) - V^M(x) > 0 \) for all \( y > y_0 \). Therefore, if \( V^M(x|y_0) - V^M(x) \) is positive, then \( V^M(x|y) - V^M(x) \) is positive for all \( y > y_0 \). But then the rationality of strategies requires that if an \( x \)-manager accepts a \( y \)-worker, he must also accept all workers of quality higher than \( y \). We have thus proven the following results.

Lemma 5. Managers’s value functions and strategies satisfy the following monotonicity conditions:

1. Managers make match decisions by comparing the value of being matched to someone forever to the value of remaining unmatched:

\[
\text{sgn}(V^M(x|y) - V^M(x)) = \text{sgn}(f(x,y) - V^M(x));
\]

2. Managers’s acceptance strategies are monotonically increasing in workers’ quality:

\[
\alpha^M(x,y) = 1 \Rightarrow \text{for all } y' > y, \alpha^M(x,y') = 1.
\]

The result above already shows that managers will employ cutoff strategies; that is, their acceptance sets will be intervals with an upper bound of 1. Combined with the continuity of \( f \) and the fact that \( f(x,y) > 0 \), we can strengthen this result by showing that these intervals are nonempty, nondegenerate and closed, and that the lower limits of the intervals are defined by an indifference condition:

Lemma 6. Managers’s acceptance sets are closed, nonempty, and nondegenerate intervals: for all \( x \), there exists an \( a^M(x)\epsilon[0,1] \) such that \( (A^M(x)) = [a^M(x),1] \). Managers who do not accept all workers are indifferent between hiring a marginal worker and remaining without a worker:

\[
a^M(x) \neq 0 \Rightarrow V^M(x) = V^M(x|a^M(x)).
\]

Proof. See Appendix.

We now turn to the worker’s side. The first observation here is that a matched worker’s value function is strictly increasing in the manager’s quality. This is straightforward, since a higher-quality teammate increases the immediate payoff from being matched without affecting the opportunities for future upgrade. Since managers cannot separate from workers, this means that employing a higher-quality manager is an unambiguous improvement for a worker. Stating this formally:
Lemma 7. The matched worker’s value function $V^W(x \mid y)$ is strictly increasing in the manager’s quality $y$.

Proof. Suppose $V^W(x \mid y_1) \geq V^W(x \mid y_2)$ for $y_1 < y_2$. Then $f(x, y_1) < f(x, y_2)$. Together with $V^W(x \mid y_1) \geq V^W(x \mid y_2)$ this implies that the right-hand side of (13) is greater for $y_2$ than for $y_1$, so that the left-hand side must be too, i.e., $V^W(x \mid y_2) > V^W(x \mid y_1)$. Contradiction.

An immediate corollary from this lemma is that any worker will leave its current manager whenever it meets one with higher quality:

Lemma 8. Matched workers always upgrade when possible: $A^W(x \mid y) = (y, 1]$ for each $x$ and $y$.

Proof. Follows immediately from Lemma 7 and the rationality of strategies (condition 3 in the definition of equilibrium).

Next, note that unmatched workers do not lose anything by accepting a match with anyone: they obtain the immediate rewards of being matched to someone, and their options for future matches are in no way affected. It therefore follows that unmatched workers will always accept a match with anyone they meet.

Lemma 9. Unmatched workers always accept everybody: $A^W(x) = [0, 1]$ for all $x$.

Proof. See Appendix.

We now have a complete characterization of all agents’ strategies: managers use cutoff acceptance strategies, with the lower limit determined by an indifference condition, whereas workers accept everyone when unmatched and upgrade whenever possible.

5.3 Matching Patterns

We are now prepared to tackle the main question: who matches with whom in the asymmetric-separation model? To do so, we first make the simple observation that, just as in the baseline model, the matching sets are fully determined by unmatched agents’ acceptance sets. The matching density for the pair $(x, y)$ is positive if and only if $x$ and $y$ are acceptable to each other when unmatched. The intuition is straightforward: there is positive inflow to $(x, y)$ matches if and only if $x$ and $y$ accept each other, and there is positive outflow if and only if $\mu(x, y) > 0$. The two must be equal in steady state.

Lemma 10. The matching set for each worker $x$ equals the set of managers $y$ such that $x$ and $y$ are mutually acceptable to each other when unmatched: $\mu(x, y) > 0$ if and only if $\alpha^M(x, y) = \alpha^W(y, x) = 1$. 
Proof. Let \( \mu(x, y) > 0 \). Equation (19) immediately implies that \( \alpha^W(y, x) = 1 \). Furthermore, since \( D(x, y) \geq \delta > 0 \) for all \( x \) and \( y \), the left-hand side of equation (18) is positive. For the right-hand side to be positive, we require \( \alpha^M(x, y) = 1 \).

Now let \( \alpha^M(x, y) = \alpha^W(y, x) = 1 \). Notice that since the type density is everywhere positive \((l > 0)\) and all matches are dissolved at a positive rate \( \delta > 0 \), \( u^M \) and \( u^L \) are also everywhere positive. Thus \( W(x, y) \geq \rho u^W(y) \alpha^W(y, x) > 0 \), and the right-hand side of (18) is positive. For the left-hand side to be positive, we need \( \mu(x, y) > 0 \).

The problem of describing the matching sets therefore reduces to describing the acceptance sets of unmatched agents. Recalling the results from the strategy section above, we see that the only missing piece in the puzzle is a characterization of the acceptance thresholds of managers, \( a^M(x) \).

We begin by rewriting the value functions using the findings in the previous section. In particular, plugging the surplus equation (20) into the unmatched managers’s value function (16), applying Lemma 6, integrating, collecting terms, and noting that \( a^M(x) \) must be chosen optimally, yields

\[
V^M(x) = \max_{a^M(x)} \frac{\int_{a^M(x)}^1 H(x, y) f(x, y) \, dy}{1 + \int_{a^M(x)}^1 H(x, y) \, dy}, \quad \text{where} \quad H(x, y) = \frac{\Omega(x, y)}{\tau + D(x, y)}. \tag{21}
\]

Applying Lemmas 8 and 9 to (14) and (15), the match dissolution and opportunity rates simplify to

\[
\Omega(x, y) = \rho \left( u^W(y) + \int_0^x \mu^W(y, x') \, dx' \right); \tag{22}
\]

\[
D(x, y) = \delta + \rho \int_x^1 u^M(x', y) \alpha^M(x', y) \, dx'. \tag{23}
\]

It is easy to see that the opportunity rate \( \Omega \) increases in \( x \), while the dissolution rate \( D \) decreases. Consequently, the ratio \( H \) is monotonically increasing, and hence almost everywhere differentiable as a function of \( x \). It then follows that the value function \( V \) is also almost everywhere differentiable. More precisely, these observations give us the following result:

**Lemma 11.** The unmatched managers’s value function \( V^M(x) \) is almost everywhere differentiable. Furthermore, \( H_1(x, y) \) is defined almost everywhere, with \( H_1(x, y) > 0 \) whenever \( y > a^M(x) \).

**Proof.** Because the integrands in (22) and (23) are non-negative, \( \Omega(x, y) \) is weakly increasing in \( x \) and \( D(x, y) \) is weakly decreasing in \( x \). Therefore, for each \( y \), these functions are almost everywhere differentiable as functions of \( x \), with \( \Omega_1(x, y) \geq 0 \) and \( D_1(x, y) \leq 0 \). Consequently, \( H(x, y) \) is weakly increasing in \( x \), and \( H_1(x, y) \) exists almost everywhere, with \( H_1 \geq 0 \). This, in turn, implies that \( V(x) \) is also differentiable almost everywhere (by the Envelope Theorem).

For the final statement, note that \( D_1(x, y) = -\rho u^M(x) \alpha^M(x, y) < 0 \) whenever \( y > a^M(x) \). Thus it is also true that \( H_1(x, y) > 0 \) for all \( y > a^M(x) \). \( \square \)
We can now proceed to our key observation: the managers’s acceptance threshold function $a^M(x)$ is strictly increasing whenever payoffs are weakly log-supermodular. The proof proceeds along the same lines as the proof of Proposition 3 in Smith [2006]. The key difference from Smith’s case is that asymmetric separation adds an additional benefit to higher-quality managers: they have a lower probability of being fired or let go. This additional benefit (as captured by $H_1(x, y) > 0$) is sufficient to make higher-quality managers strictly more selective than lower-quality managers, even when payoffs are only weakly log-supermodular. The following lemma is proved in the Appendix.

**Lemma 12.** When the payoff function $f(x, y)$ is weakly log-supermodular, the managers’s acceptance threshold function $a^M(x)$ is strictly increasing at all $x$ such that $a^M(x) > 0$.

We now have a complete characterization of all acceptance sets, and thus also of matching sets. In particular, we have obtained a sufficient condition for strict setwise PAM (recall Definition 4) in this model. That is, matching is strictly positively assortative setwise whenever payoffs are weakly log-supermodular. This result follows from the fact that matching sets are completely determined by managers, whose thresholds are increasing by Lemma 12.

**Proposition 3.** The asymmetric separation model exhibits strict setwise PAM whenever payoffs are weakly log-supermodular.

**Proof.** Let payoffs be weakly log-supermodular. By Lemma 9, $\alpha^W(y, x) = 1$ everywhere. By Lemma 6, $\alpha^M(x, y) = 1$ if and only if $y \geq a^M(x)$. Then, by Lemma 10, $M^M(x) = [a^M(x), 1]$ for all $x$. Since $a^M(x)$ is strictly increasing whenever it is not zero (by Lemma 12), it follows that matching is strictly positively assortative (see the note after Definition 4).

In particular, this result implies that the block segregation pattern from the baseline model is not robust to asymmetric separation. With asymmetric separation, weak supermodularity (and thus also multiplicative separability) of payoffs is sufficient for strict setwise PAM (which rules out block segregation). Note, however, that the asymmetric separation result is considerably weaker than the symmetric separation result. With symmetric separation, the matching pattern converges to perfect PAM, whereas asymmetric separation gives us only strict setwise PAM.

6 Conclusion

Search and matching theory has heretofore ignored the very real and important element of separation. Separation is undoubtedly an important feature of firms today. If both sides can search and upgrade while matched, we find that there exist rather weak conditions on the primitives of the model\textsuperscript{13} that ensure the existence of an equilibrium that approaches perfect positively assortative

\textsuperscript{13}In particular, regardless of other parameter values, there always exists a sufficiently high interest rate $r$ such that the conditions are satisfied.
matching, as the external match dissolution rate decreases. When only one side of the market can upgrade, it is not possible to achieve convergence to perfect PAM, but strict setwise PAM obtains even with weakly log-supermodular payoffs.

Block segregation is inefficient because agents match with the “wrong” partner. When matching in blocks, every agent is comfortable matching with an agent of similar, but not identical, quality. Even though the agent prefers to match with someone identical, the cost of waiting and the permanence of the match causes the agent to relax his standards and agree to match with agents within a neighborhood of quality. But if separation is possible, agents can still garner the benefits of matching without the cost of waiting. Now, the agents can simply match early and wait until they meet another agent of quality more similar than their current match. When they find that better match, they will dissolve their prior contract and form a new one. As this proceeds, eventually all agents will match with their identical counterpart. This results in positive assortative matching, which maximizes surplus. Thus, allowing separation increases efficiency because it prevents matching with the “wrong” agent, as happens when agents match in blocks when they lack this option to separate.

Future research in this area will proceed in both theoretical and empirical directions. On the theoretical level, the question remains about the effects of intramatch bargaining on matching outcomes. In practice, agents do not decide instantaneously to separate, but rather go through an extended bargaining game in which the party threatening to leave may solicit concessions from the other party. The matching model still has a bright future ahead for organizational theory.

Appendix

Proof of Lemma 1. If some unmatched $x$-agent of supertype $i$ does not strictly prefer accepting a potential partner $y$ to remaining unmatched, it must be the case that $V^i(x) \geq V^i(x | y)$. Equation (7) then implies

$$V^i(x) \geq V^i(x | y) \geq f(x, y) + \frac{1}{r} \int_0^1 \max (V^i(x | y') - V^i(x), 0) \Omega^i(x, y') dy'.$$

The last term in the expression above is just $V^i(x)$ (by (6)). We thus obtain

$$V^i(x) \geq f(x, y) + V^i(x),$$

which is impossible, since $f(x, y) > 0$ everywhere. Contradiction.

Proof of Lemma 2. By Lemma 1, $V^i(x | y) > V^i(x)$ for all $x$ and $y$. This implies that the second summand in the right-hand-side of the definition of $V^i(x | y)$ (eq. (7)) is negative. Furthermore, the integrand in the third summand is no more than $\max(V^i(x | y) - V^i(x), 0)$, so that the integral
is no more than \(V^i(x)\) as defined in equation (6). Thus the entire right hand side is less than \(f(x, y) + V^i(x)\). Thus (7) implies \(V^i(x | y) - V^i(x) < f(x, y)\).

**Proof of Lemma 3.** Let \(g(x, y) \equiv f(x, y) + \frac{1}{r}(V^i(x) - V^i(x | y))(\delta + D^{-i}(y, x))\).

First notice that \(V^i(x | y)\) is increasing in \(y\) if and only if \(g(x, y)\) is. For suppose \(V^i(x | y_1) \geq V^i(x | y_2)\), while \(g(x, y_1) < g(x, y_2)\). Then, using equation (7) for \(y_1\), we obtain

\[
V^i(x | y_1) = g(x, y_1) + \frac{1}{r} \int_0^1 \max (V^i(x | y') - V^i(x | y_1), 0) \Omega(y') dy' < g(x, y_2) + \frac{1}{r} \int_0^1 \max (V^i(x | y') - V^i(x | y_2), 0) \Omega(y') dy' = V^i(x | y_2),
\]

RAA. It follows that \(\text{sgn}(V_2^i(x | y)) = \text{sgn}(g_2(x, y))\).

Taking the derivative of \(g\) with respect to \(y\), we obtain

\[
g_2(x, y) = f_2(x, y) + \frac{1}{r}(V^i(x) - V^i(x | y))D^{-i}_1(y, x) - \frac{1}{r}V_2^i(x | y)(\delta + D^{-i}(y, x)).
\]

Since \(\text{sgn}(V_2^i(x | y)) = \text{sgn}(g_2(x, y))\) and \(\delta + D^{-i}(y, x) > 0\), it follows that

\[
\text{sgn}(V_2^i(x | y)) = \text{sgn}(g_2(x, y)) = \text{sgn} \left( f_2(x, y) + \frac{1}{r}(V^i(x) - V^i(x | y))D^{-i}_1(y, x) \right).
\]

**Proof of Theorem 1.** Suppose everyone plays the conjectured “accept when unmatched, upgrade when possible” strategy. Our first, and most difficult, task is to show that this strategy profile induces a well-defined steady-state match distribution. We begin by noting that the conjectured strategies transform the opportunity and separation rate equations (4) and (5) into

\[
\Omega^i(x, y) = \rho \left( u^{-i}(y) + \int_0^x \mu^{-i}(y, x') dx' \right) = \rho \left( l(y) - \int_x^1 \mu^{-i}(y, x') dx' \right);
\]

\[
D^i(x, y) = \rho \int_y^1 \left( u^{-i}(y') + \int_0^x \mu^{-i}(y', x') dx' \right) dy' = \rho \int_y^1 \left( l(y') - \int_x^1 \mu^{-i}(y', x') dx' \right) dy'.
\]

Inserting these values in the steady-state equation (8) and simplifying, we obtain

\[
\mu(x, y) \left\{ \delta + \rho \left[ \int_y^1 \left( l(y') - \int_x^1 \mu(x', y') dx' \right) dy' + \int_x^1 \left( l(x') - \int_y^1 \mu(x', y') dy' \right) dx' \right] \right\} = \rho \left( l(y) - \int_x^1 \mu(x', y) dx' \right) \left( l(x) - \int_y^1 \mu(x, y') dy' \right).
\]

30
Let $\mathcal{A}$ be the space of admissible match densities. That is, $\mathcal{A}$ consists of all integrable functions $\mu : [0, 1]^2 \to \mathbb{R}_+$ that satisfy, for each $x \in [0, 1]$, $\int_0^1 \mu(x, y) \, dy \leq l(x)$ and $\int_0^1 \mu(y, x) \, dy \leq l(x)$. Let $\phi \equiv \delta/\rho$. Define the operator $T$ on $\mathcal{A}$ as mapping $\mu$ to the function $T\mu : [0, 1]^2 \to \mathbb{R}_+$ such that, for any $x$ and $y$

$$T\mu(x, y) = \frac{\left(l(y) - \int_x^1 \mu(x', y) \, dx'\right) \left(l(x) - \int_y^1 \mu(x, y') \, dy'\right)}{\phi + \int_y^1 \left(l(y') - \int_x^1 \mu(x', y') \, dx'\right) \, dy' + \int_x^1 \left(l(x') - \int_y^1 \mu(x', y') \, dy'\right) \, dx'}.$$

(24)

Note that $T\mu$ is defined everywhere, since $\phi > 0$ and all other terms are non-negative by admissibility of $\mu$.

The match densities consistent with the conjectured strategies are precisely the fixed points of $T$. The standard approach would be to endow $\mathcal{A}$ with an appropriate topology and show that $T$ is a contraction with respect to that topology, which would prove that $T$ has a unique fixed point. Unfortunately, however, $T$ does not even map $\mathcal{A}$ to itself (consider, for example, the case when $l(x) = l(y) = 1$ everywhere, $\mu(x, y) = 0$ everywhere, and $\phi = 0.1$). We therefore need to employ a more complicated solution technique.

**Discretized problem and equilibrium match densities**

We will find a fixed point of $T$ by discretizing the problem, explicitly solving the discretized problem recursively, and then showing that the unique solutions of a sequence of discretized problems converge to the unique solution of the original, continuous, problem.

For a given $K \in \mathbb{N}$, consider $2^K$ discrete types, indexed by $0$ through $2^K - 1$. Let $h_K = 1/2^K$. Consider an equally spaced grid of radius $h_K$ on $[0, 1]$, and let the $i$th type correspond to the grid point $1 - ih_K \in [0, 1]$. For any $x \in [0, 1]$, let $i_K(x)$ be the grid point closest to $x$:

$$i_K(x) = \arg\min_i |1 - ih_K - x|.$$

For any $0 \leq i < 2^K$, define $l_i^K = l(1 - ih_K)$. Finally, for a given function $\mu$ and any $0 \leq i, j < 2^K$, define

$$m_{ij, \mu}^K \equiv \mu(1 - ih_K, 1 - jh_K).$$

We can now define an operator $\hat{T}_K$ on $\mathcal{A}$ by letting $\hat{T}_K\mu$ be the function that is defined as follows. Let

$$\hat{T}_K\mu(1 - ih_K, 1 - jh_K) =$$

$$\frac{\left(l_j^K - \sum_{i' \leq i} m_{ij', \mu}^K h_K\right) \left(l_i^K - \sum_{j' \leq j} m_{ij', \mu}^K h_K\right)}{\phi + \sum_{j' < j} \left(l_j^K h_K - \sum_{i' \leq i} m_{ij', \mu}^K h_K^2\right) + \sum_{i < i'} \left(l_i^K h_K - \sum_{j' \leq j} m_{ij', \mu}^K h_K^2\right)}.$$

31
for all $0 \leq i, j < 2^K$, and let $\hat{T}_K(x, y)$ for all other $(x, y)$ be determined by linear spline interpolation from the values of $\hat{T}_K$ on the grid points $(1 - ih_K, 1 - jh_K)$.

Note that $\mu$ is a fixed point of $\hat{T}_K$ if and only if the values of $\mu$ off the grid are obtained by linear spline interpolation from the values of $\mu$ on the grid, and the values on all grid points, $0 \leq i, j < 2^K$, satisfy

$$m_{ij,\mu}^K = \frac{\left((t_j^K - \sum_{i' \leq i} m_{i'j,\mu}^K h_K^2) (t_i^K - \sum_{j' \leq j} m_{ij',\mu}^K h_K^2)\right)}{\phi + \sum_{j' < j} \left(t_j^K h_K - \sum_{i' \leq i} m_{i'j',\mu}^K h_K^2\right) + \sum_{i' < i} \left(t_i^K h_K - \sum_{j' \leq j} m_{ij',\mu}^K h_K^2\right)}.$$

(25)

Note that this equation represents the steady state in a model with $2^K$ discrete types. When all types follow the conjectured “accept anyone, upgrade when possible” strategy, the mass of the types is symmetric due to the symmetry of the solution to (25). We thus have the following result:

The discrete steady-state equation (25) can be easily solved for $m_{ij,\mu}^K$ recursively, starting with $i = j = 0$. It is trivial to check that there is exactly one admissible solution $\hat{m}_{ij}^K$ (satisfying $l_i^K \geq \sum_{j' \leq j} \hat{m}_{ij'}^K h_K^2$ and $l_j^K \geq \sum_{i' \leq i} \hat{m}_{ij}^K h_K$ for all $i$ and $j$) and that this solution is symmetric, satisfying $\hat{m}_{ij}^K = \hat{m}_{ji}^K$ for all $i$ and $j$.

Since the values of $\hat{T}_K$ on the grid fully determine the values of $\hat{T}_K$ everywhere, we have proven that $\hat{T}_K$ has a unique fixed point on $A$, namely, the function $\hat{\mu}_K$ obtained from the unique solution to (25), $\{\hat{m}_{ij}^K\}$, by linear spline interpolation. Note that $\hat{\mu}_K$ is continuous by definition and symmetric due to the symmetry of the solution to (25). We thus have the following result:

**Lemma 13.** For each $K$, $\hat{T}_K$ has a unique fixed point $\hat{\mu}_K$ on $A$. The solution is symmetric ($\hat{\mu}_K(x, y) = \hat{\mu}_K(y, x)$ for all $x$ and $y$) and everywhere continuous on $[0, 1]^2$. For each $0 \leq i, j < 2^K$, $\hat{\mu}_K(1 - ih_K, 1 - jh_K) = \hat{m}_{ij}^K$, where $\{\hat{m}_{ij}^K\}$ is the unique admissible solution to (25).

Our next step is to observe that the sequence of fixed points $\hat{\mu}_K$ converges to a limit $\mu \in A$ as $K$ goes to infinity. Let $\|\bullet\|_\infty$ denote the $L_\infty$ norm on $A$. Then we require the following Lemma.

**Lemma 14.** There exists $\mu \in A$ such that

$$\lim_{K \to \infty} \|\hat{\mu}_K - \mu\|_\infty = 0.$$

$\mu$ is symmetric and everywhere continuous on $[0, 1]$.

**Proof of Lemma 14.** We first show that the values of $\hat{\mu}_K$ and $\hat{\mu}_{K+1}$ evaluated on the grid corresponding to $K$ get closer and closer to each other as $K$ grows large. To see this, take any $K$ and consider the grids $K$ and $K + 1$. Note that $K$ is a subgrid of $K + 1$, with the $(i, j)$ node of $K$ corresponding to the $(2i, 2j)$ node of $K + 1$. Also observe that, as $K \to \infty$,

$$e^K \equiv \max_{i, j < 2^K} |\hat{m}_{ij+1}^K - \hat{m}_{ij}^K| \to 0.$$
\[ \hat{\epsilon}^K \equiv \max_{i < 2^k} |l(1 - ih_K) - l(1 - (i + 1)h_K)| \to 0. \]

Rewriting the discrete steady state equation (25) for \( K \) and \( K + 1 \) and grouping neighboring terms in \( K + 1 \) together, we can show by induction on \( i \) and \( j \) that

\[ \left| \hat{m}^K_{ij} - \hat{m}^{K+1}_{2i,2j} \right| = O \left( \max\{\hat{\epsilon}^K, \epsilon^K, h^K\} \right) \]

(details available on request). Since \( \epsilon^K, \hat{\epsilon}^K, \) and \( h^K \) all vanish as \( K \to \infty \), this shows that

\[ \lim_{K \to \infty} \max_{i,j} \left| \hat{m}^K_{ij} - \hat{m}^{K+1}_{ij} \right| = 0. \]

Next, since \( \hat{\mu}_K \) is obtained by a linear spline from its values on the grid, we know that as \( K \) becomes infinite and the grid points get close to each other, the difference between \( \hat{\mu}_K \) on any point and on its closest grid point becomes negligible:

\[ \lim_{K \to \infty} \max_{(x,y) \in [0,1]^2} \left| \hat{\mu}_K(x,y) - \hat{\mu}_K(\iota_K(x), \iota_K(y)) \right| = 0. \]

Since the values of \( \hat{\mu}^K \) and \( \hat{\mu}^{K+1} \) are close to each other on the grid, and values of these two functions elsewhere are close to their grid values, the triangle inequality implies that the values of the two functions are close to each other everywhere, i.e.,

\[ \lim_{K \to \infty} \max_{(x,y) \in [0,1]^2} \left| \hat{\mu}_K(x,y) - \hat{\mu}_{K+1}(x,y) \right| = 0. \]

That is, the sequence of \( \hat{\mu}_K \) is Cauchy in the norm \( L_\infty \). \( \mathcal{A} \) is clearly complete with this norm, and the result of Lemma 14 follows.

The key observation in the proof of Lemma 14 is that the solutions to (25) for successive values of \( K \) become increasingly close to each other, so that \( \{\hat{\mu}_K\} \) is a Cauchy sequence and therefore converges in the complete space \((\mathcal{A}, L_\infty)\).

All that remains to be proven is that \( \mu \) is a fixed point of \( T \). The proof of this fact hinges on the observation that \( \hat{T}_K \) is a quadrature approximation of \( T \) on the \( K \)-grid and thus converges to \( T \) on \( \mathcal{A} \). Continuity properties of the functions involved, together with the facts that \( \lim_{K \to \infty} \hat{\mu}_K = \mu \) and \( \hat{\mu}_K \) is a fixed point of \( \hat{T}_K \), yield the desired result:

**Lemma 15.** \( T\mu = \mu \).

**Proof of Lemma 15.** We begin by observing that both \( T \) and \( \hat{T}_K \) for any \( K \) are Lipschitz with respect to the \( L_\infty \) norm on \( \mathcal{A} \).

Next, consider any bounded and uniformly continuous function \( \phi : [0,1]^2 \to \mathbb{R}_+ \). For any fixed \((x,y)\), standard results on the convergence of quadrature estimates to the true values of integrals
yield \( |T\phi(x, y) - \hat{T}_K\phi(x, y)| \to 0 \) as \( K \to \infty \). Since \( \phi \) is uniformly continuous and \( T \) and \( \hat{T}_K \) are Lipschitz, hence also uniformly continuous, this also implies that \( \|T\phi - \hat{T}_K\phi\|_\infty \to 0 \) as \( K \to \infty \).

Next, observe that \( \mu \) is bounded (by 0 below and \( l^2 \rho/\delta \) above) and continuous, hence also uniformly continuous on \([0, 1]\). Thus \( \|T\mu - \hat{T}_K\mu\|_\infty \to 0 \) as \( K \to \infty \).

Furthermore, \( \|\hat{T}_K\mu - \hat{\mu}_K\|_\infty = \|\hat{T}_K\mu - \hat{T}_K\hat{\mu}_K\|_\infty \), since \( \hat{\mu}_K \) is a fixed point of \( \hat{T}_K \). Since \( \hat{T}_K \) is Lipschitz, there exists \( z \) such that the left hand side is no more than \( z\|\mu - \hat{\mu}_K\|_\infty \), which goes to zero as \( K \to \infty \) by Lemma 14. Thus \( \|\hat{T}_K\mu - \hat{\mu}_K\|_\infty \to 0 \) as \( K \to \infty \).

Finally, \( \|\hat{\mu}_K - \mu\|_\infty \to 0 \) as \( K \to \infty \) by Lemma 14.

By the triangle inequality, \( \|T\mu - \mu\|_\infty \leq \|T\mu - \hat{T}_K\mu\|_\infty + \|\hat{T}_K\mu - \hat{\mu}_K\|_\infty + \|\hat{\mu}_K - \mu\|_\infty \). Since each of the terms on the right goes to zero as \( K \) goes to infinity, we conclude that the left-hand side must be zero, i.e., \( T\mu = \mu \).

\( \square \)

**Value functions in conjectured equilibrium**

The result above shows that the conjectured equilibrium strategies give rise to a well-defined, symmetric match density function. Since there is symmetry in both strategies and match distributions, the value function equations (6) and (7) are the same for managers and workers. Thus, if value functions are well defined, they are symmetric, and we can drop the \( M \) and \( W \) superscripts. Rewriting and merging the value equations, we see that values are fully defined by solutions \( V(x \mid y) \) to the following equation:

\[
V(x \mid y) = \varphi_1(x, y) + \int_0^1 V(x \mid y')\varphi_2(x, y, y') dy' + \int_y^1 V(x \mid y')\varphi_3(x, y, y')) dy',
\]

where the \( \varphi_i \) are bounded and continuous functions that are differentiable in \( y \) and are derived from the conjectured strategies via the steady state matching densities. Thus, for each \( x \), \( V(x \mid y) \) is given by a linear Volterra-Fredholm integral equation on a compact domain, where all kernels are bounded, continuous, and differentiable in \( y \). Standard results in the theory of integral equations imply that a unique solution \( V(x \mid y) \) exists, and that it is bounded, continuous, and differentiable with respect to \( y \).

\( \square \)

**Proof of Theorem 2.** We begin by looking at the discretized model for some \( K \).

First, let us show that \( L_i^K - \hat{M}_{ii}^K = \mathcal{O}(\delta^{2-(K+1)}) \) for all \( i \leq K \). We do this by using induction on \( n \) to prove the statement that \( L_i^K - \hat{M}_{ii}^K = \mathcal{O}(\delta^{2-(n+1)}) \) for all \( i \leq n \).

The basis is straightforward: \( \hat{M}_{00}^K = \rho(L_0^K - \hat{M}_{00}^K)^2 \). Since \( \hat{M}_{00}^K \) is bounded above by \( L_0^K \), we see immediately that the LHS is \( \mathcal{O}(\delta) \). It follows that \( L_0^K - \hat{M}_{00}^K = \mathcal{O}(\delta^{1/2}) \).

Now suppose \( L_i^K - \hat{M}_{ii}^K = \mathcal{O}(\delta^{2-n}) \) for all \( i \leq n - 1 \). Then, in particular, \( \hat{M}_{ij}^K = \mathcal{O}(\delta^{2-n}) \) for all \( i \leq n - 1 \) and \( j \neq i \). By symmetry of \( M \), \( \hat{M}_{ji}^K = \mathcal{O}(\delta^{2-n}) \) for all \( i \leq n - 1 \) and \( j \neq i \). Then also

\[34\]
\[ \sum_{i' < n} \hat{M}_{i' n}^K = \mathcal{O}(\delta^{2-n}) \] and \( \sum_{i' < n} \hat{M}_{n i'}^K = \mathcal{O}(\delta^{2-n}) \). Finally,

\[
\sum_{j' < n} \left( L_{j'}^K - \sum_{i' \leq n} \hat{M}_{i' j'}^K \right) = \sum_{j' < n} \left( L_{j'}^K - \hat{M}_{j' j'}^K \right) + \mathcal{O}(\delta^{2-n}) = \mathcal{O}(\delta^{2-n})
\]

by the induction assumption, and

\[
I_n^K - \sum_{i' < n} \hat{M}_{i' n}^K - \hat{M}_{n n}^K = I_n^K - \hat{M}_{n n}^K + \mathcal{O}(\delta^{2-n}).
\]

The SS equation thus implies that

\[
\rho(I_n^K - \hat{M}_{n n}^K + \mathcal{O}(\delta^{2-n}))^2 = \mathcal{O}(\delta^{2-n}),
\]

which in turn shows that \( I_n^K - \hat{M}_{n n}^K = \mathcal{O}(\delta^{2-(n+1)}) \), or \( I_n^K - \hat{m}_{n n}^K h_K = \mathcal{O}(\delta^{2-(n+1)}) \), which completes the induction.

Now, pick any \( R, \epsilon > 0 \). By the triangle inequality,

\[
\left\| I(x) - \int_{x-R}^{x+R} \mu(x, y) \, dy \right\|_{\infty} \leq \left\| I(x) - I_{i_k(x)}^K \right\|_{\infty} + \left\| I_{i_k(x)}^K - \sum_{1-i h_k \in (x-R, x+R)} \hat{m}_{i_k(x), i h_k}^K h_K \right\|_{\infty} + \left\| \sum_{1-i h_k \in (x-R, x+R)} \hat{m}_{i_k(x), i h_k}^K h_K - \int_{x-R}^{x+R} \mu(x, y) \, dy \right\|_{\infty}\]

(26)

By our previous results on the convergence of the discrete to the continuous model, there exists \( K_0 > 0 \) such that the first and last terms on the right-hand side of (26) are less than \( \epsilon/4 \) for each \( K \geq K_0 \). Note also that \( 1 - i_k(x) h_K \to x \) as \( K \to \infty \), so that there exists \( K_1 \) such that \( 1 - i_k(x) h_K \in (x-R, x+R) \) for all \( K \geq K_1 \). Let \( \bar{K} = \max\{K_0, K_1\} \).

By the result we just proved inductively, \( I_{i_k(x)}^K - \hat{m}_{i_k(x), i h_k}^K h_K = \mathcal{O}(\delta^{2-(K+1)}) \). Since \( 1 - i_k(x) h_K \in (x-R, x+R) \), this implies that the second term on the right-hand side of (26) is \( \mathcal{O}(\delta^{2-(K+1)}) \). Since the first and the last terms are each less than \( \epsilon/4 \), this implies that there exists \( \delta_0 > 0 \) such that the entire expression is less than \( \epsilon \) for all \( \delta < \delta_0 \). That is,

\[
\lim_{\delta \to 0^+} \left\| I(x) - \int_{x-R}^{x+R} \mu(x, y) \, dy \right\|_{\infty} = 0.
\]

\[\square\]

Proof of Lemma 6. First observe that if \( f(x, 0) \geq V^M(x) \), then, by Lemma 5, \( A^M(x) = [0, 1] \). Now, let \( f(x, 0) < V^M(x) \).

Suppose \( f(x, y) < V^M(x) \) for all \( y \in [0, 1] \). Then, by Lemma 5, \( A^M(x) \subset \{1\} \), so that, by equation (16), an \( x \)-manager’s value of being unmatched would be zero. But then, since \( f \)
is everywhere positive, \( f(x, y) > 0 = V^M(x) \) for all \( y \in [0, 1] \). Then Lemma 5 implies that \( A^M(x) = [0, 1] \), which contradicts \( A^M(x) \subset \{1\} \). RAA.

Hence (\( \exists y \in [0, 1] \))(\( f(x, y) \geq V^M(x) \)). Since \( f \) is continuous and \( f(x, 0) < V^M(x) \), the Intermediate Value Theorem implies that

\[
(\exists a^M(x) \in (0, 1))(f(x, a^M(x)) = V^M(x)).
\]

By Lemma 5, \( V^M(x|a^M(x)) = V^M(x) \) and \( A^M(x) = [a^M(x), 1] \).

**Proof of Lemma 9.** Since \( V^W(x|y) \) is increasing in \( y \) by Lemma 7, we only need to show that \( V^W(x|0) \geq V^W(x) \). Suppose not, i.e., \( V^W(x|0) < V^W(x) \). Then we have

\[
\int_0^1 (V^W(x|y') - V^W(x|0), 0) u^M(y')\alpha^M(y', x) dy' \\
\int_0^1 (V^W(x|y') - V^W(x), 0) u^M(y')\alpha^M(y', x) dy' = V^W(x),
\]

where the equality follows by the definition of \( V^W(x) \) (equation (12)).

Now, look at the right-hand side of the definition of \( V^W(x|0) \) (equation (13)). We have just shown that the last term there is no less than \( V^W(x) \). Furthermore, the first term, \( f(x, 0) \) is positive by definition, and the second term, \( (\delta/r)(V^W(x) - V^W(x|y)) \) is positive by assumption. But then the entire right-hand side of (13) is greater than \( V^W(x) \). Therefore, the same must be true of the left-hand side: \( V^W(x|0) > V^W(x) \). RAA.

**Proof of Lemma 12.** Take any \( x \) such that \( a^M(x) > 0 \). By the Envelope Theorem and (21),

\[
V^{M'}(x) = \frac{\left( \int H_1 f + \int H f_1 \right)(1 + \int H) - (\int H f)(\int H_1)}{(1 + \int H)^2},
\]

where all integrals are taken with respect to \( dy \) over the interval \([a^M(x), 1]\); we have omitted the arguments \((x, y)\) of all functions for brevity.

By Lemmas 5 and 6, \( f(x, y) > V^M(x) \) for all \( y > a^M(x) \). Furthermore, by Lemma 11, \( H_1(x, y) > 0 \) for \( y > a^M(x) \) where the derivative exists. Therefore,

\[
\int H_1 f > V^M(x) \int H_1 = \frac{\int H f}{1 + \int H} \int H_1,
\]

so that \( V^{M'}(x) > (\int H f_1)/(1 + \int H) \) and

\[
\frac{V^{M'}(x)}{V^M(x)} > \frac{\int_{a^M(x)}^1 H(x, y)f_1(x, y) dy}{\int_{a^M(x)}^1 H(x, y)f(x, y) dy} > \frac{f_1(x, a^M(x))}{f(x, a^M(x))},
\]

where the weak inequality follows by weak log-supermodularity of \( f \).

36
Finally, note that, by Lemmas 6 and 5, \( f(x, a^M(x)) = V^M(x) \) for all \( x \) where \( a^M(x) > 0 \). Differentiating this identity yields

\[
f_1(x, a^M(x)) + f_2(x, a^M(x))a^M'(x) = V^M'(x),
\]

so that

\[
\frac{V^M'(x)}{V^M(x)} = \frac{f_1(x, a^M(x)) + f_2(x, a^M(x))a^M'(x)}{f(x, a^M(x))}.
\]

Substituting this identity into (27) yields \((f_2(x, a^M(x))/f(x, a^M(x)))a^M'(x) > 0\). Since \( f_2, f > 0 \) everywhere, we must conclude that \( a^M'(x) > 0 \). \( \Box \)
References


