

A Generalized Measure of Riskiness

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This paper proposes a generalized measure of riskiness that nests the original measures pioneered by Aumann and Serrano (Aumann, R. J., R. Serrano. 2008. An economic index of riskiness. *J. Political Econom.* **116**(5) 810–836) and Foster and Hart (Foster, D. P., S. Hart. 2009. An operational measure of riskiness. *J. Political Econom.* **117**(5) 785–814). The paper introduces the generalized options' implied measure of riskiness based on the risk-neutral return distribution of financial securities. It also provides asset allocation implications and shows that the forward-looking measures of riskiness successfully predict the cross section of 1-, 3-, 6-, and 12-month-ahead risk-adjusted returns of individual stocks. The empirical results indicate that the generalized measure of riskiness is able to rank equity portfolios based on their expected returns per unit of risk and hence yields a more efficient strategy for maximizing expected return of the portfolio while minimizing its risk.

Key words: riskiness; economic index of riskiness; operational measure of riskiness; risk-neutral measures; stock returns

History: Received August 7, 2010; accepted April 8, 2011, by Wei Xiong, finance. Published online in *Articles in Advance* June 24, 2011.

1. Introduction

Aumann and Serrano (2008) introduce a measure of riskiness based on investors' risk tolerance. They define the riskiness of a gamble as the reciprocal of the constant absolute risk aversion of an individual, implying that less-averse individuals accept riskier gambles. Foster and Hart (2009) also develop a riskiness measure that depends only on the gamble and not on the decision maker's aversion to risk. This alternative measure of riskiness determines the critical wealth level below which it becomes risky to accept the gamble. The economic index measure of riskiness pioneered by Aumann and Serrano (2008) looks for the critical utility regardless of wealth, whereas the operational measure of riskiness initiated by Foster and Hart (2009) looks for the critical wealth regardless of utility.

This paper contributes to the literature by (i) proposing a generalized measure of riskiness that encompasses the original measures of both Aumann and Serrano (2008) and Foster and Hart (2009), and (ii) introducing a model-free options' implied measure of riskiness based on the *risk-neutral* return distribution of financial securities. We find that an acceptance or rejection decision should be characterized by a utility function and a current wealth level; i.e., when measuring riskiness of an asset, one should

look for the critical wealth by taking into account the investors' risk tolerance. Hence, riskiness is defined as the minimal wealth level at which the risky asset may be accepted, but this minimal reserve needed for the risky asset is allowed to take different values depending on the risk aversion of decision makers.

Both Aumann and Serrano (2008) and Foster and Hart (2009) initiate riskiness measures based on the *physical* return distribution of gambles. However, it is extremely difficult to come up with accurate estimates under the physical measure because generating empirical measures of riskiness requires precise estimates of the mean, standard deviation, and higher-order moments of the return distribution. It is well known that computing the moments of the return distribution is a difficult task because one has to know the exact return distribution under the physical measure. Because this is not possible, one needs to make a distributional assumption, but then one needs a very long sample to generate reliable estimates of the moments under the assumed distribution. This paper makes an innovative contribution to the literature by providing a *distribution-free* generalized measure of riskiness that can be obtained from actively traded options and does not rely on any particular assumptions about the return distribution.

The paper also provides asset allocation implications of the generalized measure of riskiness. The

modern theory of portfolio choice determines the optimum asset mix by maximizing the expected risk premium per unit of risk in a mean-variance framework or the expected value of a utility function approximated by the expected return and variance of the portfolio. In both cases, investors need to predict the expected future return and standard deviation of risky assets in the portfolio. We show that the forward-looking measures of riskiness successfully predict the cross section of 1-, 3-, 6-, and 12-month-ahead risk-adjusted returns of individual stocks. Hence, one can use the generalized measure of riskiness to rank equity portfolios based on their expected returns per unit of risk and thus obtain a more efficient strategy for maximizing the expected return of the portfolio while minimizing its risk.

This paper is organized as follows. Section 2 provides the theoretical framework. Section 3 introduces the generalized measure of riskiness. Section 4 presents the forward-looking options' implied measure of riskiness. Section 5 contains the data description and variable definitions. Section 6 presents the empirical results. Section 7 concludes the paper.

2. The Model

Our model setup closely follows Foster and Hart (2009). At every period $t = 0, 1, \dots$, let \mathcal{G}_{t+1} denote the collection of gamble g_{t+1} , which is a real-valued random variable having some negative values—losses are possible—and with positive expected value, i.e., $P[g_{t+1} < 0] > 0$ and $E[g_{t+1}] > 0$. We assume that g_{t+1} is a discrete variable and takes finite values, say, x_1, x_2, \dots, x_n with respective probabilities p_1, \dots, p_n ($p_i > 0$ and $\sum_{i=1}^n p_i = 1$). Let $L[g_{t+1}] := -\min_i x_i > 0$ be the maximal loss of g_{t+1} , and let $M[g_{t+1}] := \max_i x_i > 0$ be the maximal gain of g_{t+1} . Let the decision maker's initial wealth be W_0 . At time t , the decision maker is offered a gamble $g_{t+1} \in \mathcal{G}_{t+1}$ that she may accept or reject. If she accepts g_{t+1} , her wealth at time $t + 1$ is $W_{t+1} = W_t + g_{t+1}$. If she rejects g_{t+1} , her wealth at time $t + 1$ is $W_{t+1} = W_t$.

Let G denote the process $(g_{t+1})_{t=0,1,\dots}$. We make no restrictions on the random variable g_{t+1} and no assumptions about the underlying probability distribution on the space of processes from which G is drawn. The decision maker has no information about future gambles and does not have information on how her current decision will affect future gambles. Assume that \mathcal{G}_{t+1} is generated by a finite set of gambles $\mathcal{G}_0 = \{\mathcal{G}_0^{(1)}, \mathcal{G}_0^{(2)}, \dots, \mathcal{G}_0^{(m)}\} \subset \mathcal{G}_{t+1}$.

Under this setup, Foster and Hart (2009) derive a simple rule to distinguish between gambles that are "risky" and gambles that are not. Let J denote the *critical-wealth function* that associates to each gamble $g_{t+1} \in \mathcal{G}_{t+1}$, a number $J[g_{t+1}]$ in $[0, \infty]$ where the gamble g_{t+1} is rejected at wealth W_t if $W_t < J[g_{t+1}]$ and

is accepted if $W_t \geq J[g_{t+1}]$. Let s_j denote the strategy: reject the gamble g if $W_t < J[g_{t+1}]$ and accept the gamble g_{t+1} if $W_t \geq J[g_{t+1}]$. The function J satisfies $J[\lambda g_{t+1}] = \lambda J[g_{t+1}]$ for any positive value $\lambda > 0$.

As shown in Foster and Hart (2009), the strategy s_j always guarantees no bankruptcy. Foster and Hart (2009) use the following concave function

$$\phi[\lambda] = E(\log(1 + \lambda g_{t+1}))$$

and show that for every gamble $g_{t+1} \in \mathcal{G}_{t+1}$, there exists a unique real number $R[g_{t+1}] > 0$ determined by the equation

$$E \log \left(1 + \frac{1}{R[g_{t+1}]} g_{t+1} \right) = 0, \quad (1)$$

such as a strategy $s = s_j$, with critical function J , that guarantees no bankruptcy if and only if $J[g_{t+1}] > R[g_{t+1}]$.

Consider a gamble in which the decision maker gains 120 with probability 0.5 and loses 100 with probability 0.5. Following Foster and Hart (2009), it can be immediately verified from Equation (1) that $R[g_{t+1}] = 600$, where $R[g_{t+1}]$ is solution to the equation

$$\frac{1}{2} \log \left(1 + \frac{1}{R[g_{t+1}]} 120 \right) + \frac{1}{2} \log \left(1 - \frac{1}{R[g_{t+1}]} 100 \right) = 0. \quad (2)$$

To avoid bankruptcy, the decision maker rejects the offer when her wealth is less than 600. Having all decision makers with initial wealth $W_t \geq 600$ not reject the offered gamble is a relatively rare occurrence. In general, some will prefer to reject the gamble while others will prefer not to reject the gamble.

For example, a representative investor with constant relative risk-aversion (CRRA) utility $u[W] = W^\delta / \delta$, where $\delta = 3.0813$ will reject the gamble g_{t+1} if and only if $E_t u[W_t + g_{t+1}] < u[W_t]$. This condition is equivalent to saying that

$$\frac{E_t(1 + (1/W_t)g_{t+1})^\delta - 1}{\delta} < 0. \quad (3)$$

Recognizing that the function $\nu[x] = (E_t(1 + xg)^\delta - 1) / \delta$ is concave and admits a unique solution $1/1,850$ that solves for $\nu[x] = 0$, we deduce from (3) that $1/W_t > 1/1,850$. Therefore, a representative investor with CRRA utility and an initial wealth $W_t < 1,850$ will reject the gamble g . The investor's critical wealth level is $R[g_{t+1}] = 1,850$. In other words, to determine her critical wealth level, the investor solves

$$E_t \left(\frac{(1 + g_{t+1}/R[g_{t+1}])^\delta - 1}{\delta} \right) = 0 \quad (4)$$

for $R[g_{t+1}]$. Equation (4) is equivalent to

$$E_t \left(\frac{((1 + g_{t+1}/R[g_{t+1}])^\delta - 1) / \delta}{\log(1 + g_{t+1}/R[g_{t+1}])} \right) \left(\log \left(1 + \frac{g_{t+1}}{R[g_{t+1}]} \right) \right) = 0. \quad (5)$$

Equation (5) can be simplified to

$$E_t^\delta \left(\log \left(1 + \frac{g_{t+1}}{1,850} \right) \right) = 0, \tag{6}$$

where the expectation operator $E_t^\delta[\cdot]$ is defined by the decision maker’s personalized density about the gamble outcomes:

$$\frac{dP^\delta}{dP} = \Psi_\delta[g_{t+1}], \tag{7}$$

with

$$\Psi_\delta[g_{t+1}] = \begin{cases} \frac{((1 + (1/1,850)g_{t+1})^\delta - 1)/\delta}{\log(1 + (1/1,850)g_{t+1})} & \text{if } \delta < 0, \\ 1 & \text{if } \delta = 0. \end{cases} \tag{8}$$

Solving Equation (6) would produce $\delta^* = -2.0813$. To avoid bankruptcy, the decision maker with $\delta = \delta^*$ rejects any offer below 1,850. Later we will give a complete proof of (6) and summarize our main finding in Theorem 1. For $\delta = 0$, $\Psi_\delta[g_{t+1}] = 1$ and $dP^\delta = dP$. Under $\delta = 0$, (6) reduces to (2). This example shows that there exists at least one decision maker who is willing to reject any offer below 1,850. This choice suggests that the decision maker distorts the gamble before she rejects or does not reject the gamble. This example shows that the Foster and Hart (2009) result corresponds to decision makers with $\delta = 0$. In other words, the choice made by the decision maker with $\delta \neq 0$ corresponds to the decision maker who uses the distorted gain to (not) reject the gamble. The distortion parameter δ is related to the risk aversion of investors. More intuition on δ will be provided in the following sections.

3. Introducing a New Measure of Riskiness

3.1. A Generalized Measure of Riskiness

In this section, we generalize the riskiness measure of Foster and Hart (2009) when the decision maker distorts the gamble before she makes her decision. To proceed, we consider the function

$$\phi^{(\delta)}[x] = \begin{cases} \frac{E_t(1 + xg_{t+1})^\delta - 1}{\delta} & \text{if } \delta < 0, \\ E_t(\log(1 + xg_{t+1})) & \text{if } \delta = 0. \end{cases} \tag{9}$$

In Theorem 1, we show that the *critical-wealth function* J associated to each gamble $g_{t+1} \in \mathcal{G}_{t+1}$ is a function of δ . We further refer $J^{(\delta)}$ to the *critical-wealth function* for a given value δ .

THEOREM 1. Assume $\delta \leq 0$. For every gamble $g_{t+1} \in \mathcal{G}_{t+1}$, there exists a unique real number $R_{\delta,t}[g_{t+1}] > 0$ determined by the equation

$$E_t \left(\frac{(1 + (1/R_{\delta,t}[g_{t+1}])g_{t+1})^\delta - 1}{\delta} \right) = 0, \tag{10}$$

such as a strategy $s = s_{J^{(\delta)}}$ with critical function $J^{(\delta)}$ guarantees no bankruptcy if and only if $J^{(\delta)}[g_{t+1}] > R_{\delta,t}[g_{t+1}]$.

PROOF. See Appendix A in the online appendix. \square

Theorem 1 says that the minimal wealth level $J^{(\delta)}[g_{t+1}]$ at which g_{t+1} is accepted must be $R_{\delta,t}[g_{t+1}]$ or higher. Therefore, $R_{\delta,t}[g_{t+1}]$ is the minimal wealth level at which g_{t+1} may be accepted. We refer to $R_{\delta,t}[g_{t+1}]$ as the generalized measure of riskiness of g_{t+1} . A strategy s guarantees no bankruptcy if and only if for every gamble $g_{t+1} \in \mathcal{G}_{t+1}$

$$s_t \text{ rejects } g_{t+1} \text{ at all } W_t < R_{\delta,t}[g_{t+1}]. \tag{11}$$

The condition (11) does not say when to accept the gambles, it says when to reject the gamble. The acceptance level for the gamble g_{t+1} is $J^{(\delta)}[g_{t+1}]$. Consider the previous gamble, in which the decision maker gains 120 with probability 0.5 and loses 100 with probability 0.5. The critical wealth level $R_{\delta,t}[g_{t+1}]$ increases when the distortion parameter δ decreases, and the minimal critical wealth level is obtained for $\delta = 0$. We derive the condition under which the decision maker would reject a gamble when the distortion parameter δ is known. In standard portfolio allocation problems, the decision maker is allowed to take a fraction of the gamble (e.g., the risky stocks). Theorem 1 can be used to determine the fraction of a gamble that would prevent the decision maker to go bankrupt. To see this, we allow the decision maker to take any proportion of the offered gamble. She can overcome short-term losses by taking appropriately small proportions of the offered gambles. In this case, the decision maker’s wealth is

$$W_{t+1} = W_t + \alpha_t g_{t+1}, \tag{12}$$

where α_t represents a fraction of the offered gamble that depends on the gamble g_{t+1} and the value of δ (or the investor’s risk tolerance). The decision maker chooses α_t such that $W_t \geq R_{\delta,t}[\alpha_t g_{t+1}]$. Because $R_{\delta,t}[\alpha_t g_{t+1}] = \alpha_t R_{\delta,t}[g_{t+1}]$, to avoid bankruptcy, the decision maker must choose α_t , such as $\alpha_t \leq W_t/R_{\delta,t}[g_{t+1}]$. Therefore, $W_t/R_{\delta,t}[g_{t+1}]$ is the maximum amount or fraction of the gamble that the decision maker would invest.

Two key issues need further enlightenment. First, because the riskiness measure $R_{\delta,t}[g_{t+1}]$ is greater than the maximum loss of g_{t+1} , that is, $R_{\delta,t}[g_{t+1}] > L[g_{t+1}] = -\min_i g_i$ (see Lemma 1 in the online appendix), it follows that

$$1 + \frac{\min_i g_i}{R_{\delta,t}[g_{t+1}]} = 1 - \frac{(-\min_i g_i)}{R_{\delta,t}[g_{t+1}]} = 1 - \frac{L[g_{t+1}]}{R_{\delta,t}[g_{t+1}]},$$

¹ The online appendix to this paper can be found at <http://faculty.baruch.cuny.edu/tbali>.

and

$$0 < 1 - \frac{L[g_{t+1}]}{R_{\delta,t}[g_{t+1}]} < 1. \quad (13)$$

Hence, by definition, when there is a large loss,

$$\left(1 + \frac{\min_i g_i}{R_{\delta,t}[g_{t+1}]}\right)^\delta = \left(1 - \frac{L[g_{t+1}]}{R_{\delta,t}[g_{t+1}]}\right)^\delta$$

is well defined because (13) holds. Given the data set used in this paper, we do not detect any problem in estimating the riskiness measure. Of course, with a different data set, it may become necessary to put the restriction $R_{\delta,t}[g_{t+1}] > L[g_{t+1}] = -\min_i g_i$ when estimating the riskiness measure.

Second, the model in this section assumes that the outcomes of the gambles are random variables taking values in a bounded range, that is, $L[g_{t+1}] = -\min_i g_i < +\infty$. Simple rates of return $(S_i(t, \tau) - S_i(t))/S_i(t)$ are bounded below by -1 , although log returns $\log(S_i(t, \tau)/S_i(t))$ may not be bounded when $S_i(t, \tau) = 0$. The riskiness measure formula we developed in this section applies to simple returns. It applies to log returns when the realized returns take values in a bounded range. In the empirical application, we use both simple returns and log returns and find that results are quantitatively and qualitatively similar.

3.2. Properties of the Generalized Measure of Riskiness

In this section, we show that our riskiness measure shares similar properties with the two pertinent measures proposed by Aumann and Serrano (2008) and Foster and Hart (2009).

Consider a gamble g that takes the values x_1, x_2, \dots, x_n with probabilities p_1, p_2, \dots, p_n . Define the parameter λ with $0 < \lambda < 1$ and consider the λ dilution of g , $\lambda \cdot g$. The λ dilution of a gamble g is a gamble that takes the values x_1, x_2, \dots, x_n with probabilities $\lambda p_1, \lambda p_2, \dots, \lambda p_n$ and takes the value 0 with probability $1 - \lambda$.

PROPOSITION 1. For any gambles $g, h \in \mathcal{G}$,

- (i) *Distribution:* If g and h have the same distribution, then $R_\delta[g] = R_\delta[h]$
- (ii) *Homogeneity:* $R_\delta[\lambda g] = \lambda R_\delta[g]$ for every $\lambda > 0$
- (iii) *Maximum loss:* $R_\delta[g] > L[g]$
- (iv) *Subadditivity:* $R_\delta[g + h] \leq R_\delta[g] + R_\delta[h]$
- (v) *Convexity:* $R_\delta[\lambda g + (1 - \lambda)h] \leq \lambda R_\delta[g] + (1 - \lambda)R_\delta[h]$ for every $0 < \lambda < 1$
- (vi) *Dilution:* $R_\delta[\lambda \cdot g] = R_\delta[g]$ for every $0 < \lambda < 1$
- (vii) *Independent gambles:* If g and h are independent random variables, then

$$\min\{R_\delta[g], R_\delta[h]\} \leq R_\delta[g + h] \leq R_\delta[g] + R_\delta[h]. \quad (14)$$

PROOF. See Appendix A in the online appendix. \square

Similar to the measure of Foster and Hart (2009), our measure of riskiness is monotonic (see Proposition 2): a gamble that dominates (is preferred to) another has a lower riskiness. Although the standard deviation is at times used also as a measure of riskiness, it is well known that it is not a very good measure in general. One important drawback is that it is not monotonic: a gamble with higher gains and lower losses may well have a higher standard deviation and thus be incorrectly viewed as having a higher riskiness. To show the monotonicity property, we recall the first-order and second-order stochastic dominance (Rothschild and Stiglitz 1970, 1971; Hadar and William 1969; Hanoch and Levy 1969).

First-order stochastic dominance (SD₁). A gamble g first order stochastically dominates the gamble h , gSD_1h , if there exists a pair of gambles g' and h' that are defined on the same probability space such that g and g' have the same distribution; h and h' have the same distribution; and $g' \geq h'$ and $g' \neq h'$.

Second-order stochastic dominance (SD₂). A gamble g second order stochastically dominates the gamble h , gSD_2h , if h may be obtained from “mean-preserving spreads” by replacing some of g 's values with random variables whose mean is that value.

PROPOSITION 2. Assume that δ is known. If g first order stochastically dominates h or if g second order stochastically dominates h , then $R_\delta[g] < R_\delta[h]$.

PROOF. See Appendix A in the online appendix. \square

Continuity. In Proposition 3, we derive conditions under which the riskiness measures $\{R_\delta[g_n]\}_{n=1,2,\dots}$ of a sequence of gambles $\{g_n\}_{n=1,2,\dots}$ converge in distribution to the riskiness measure $R_\delta[g]$ when $\{g_n\}_{n=1,2,\dots}$ converges to g .

PROPOSITION 3. Let $\{g_n\}_{n=1,2,\dots} \subset \mathcal{G}$ be a sequence of gambles with uniformly bounded values; that is, there exists a finite K such that $|g_n| \leq K$ for all n . If $g_n \xrightarrow{p} g \in \mathcal{G}$ and $L[g_n] \rightarrow L[g]$ as $n \rightarrow \infty$, then $R_\delta[g_n] \rightarrow R_\delta[g]$ as $n \rightarrow \infty$.

PROOF. See Appendix A in the online appendix. \square

4. The Generalized Options' Implied Measure of Riskiness

The riskiness measure derived in §3 is developed under the physical measure of the gamble. When the physical distributions of the gambles are known, the riskiness measure can be implemented. However, in the finance literature, there is no general consensus about the true distributions of returns. To avoid using the physical distribution of returns or to complement

the analysis of the riskiness measure under the physical measure, one needs to infer the riskiness measure from option prices. This procedure is similar to recovering the volatility of the market return (VIX) using option prices and to the statistical value-at-risk (VaR) approach of Ait-Sahalia and Lo (2000) using the risk-neutral distribution in place of the physical distribution to estimate VaR. To complement §3 and operationalize the riskiness measure under the risk neutral measure, we define the riskiness measure from options prices as the value $R_{\delta,t}[g_{i,t+\tau}]$ that solves

$$\frac{E_t^*(1 + g_{i,t+\tau}/R_{\delta,t}[g_{i,t+\tau}])^\delta - 1}{\delta} = 0. \tag{15}$$

Thus, the paper contributes to the existing literature by introducing the generalized options' implied measure of riskiness based on the Bakshi and Madan (2000) and Bakshi et al. (2003) spanning formula. They show that any function of the form $H(S)$ with $E[H(S)] < \infty$ can be spanned by a collection of call and put options:

$$H[S] = H[\bar{S}] + (S - \bar{S})H_s[\bar{S}] + \int_{\bar{S}}^\infty H_{SS}[K](S - K)^+ dK + \int_0^{\bar{S}} H_{SS}[K](K - S)^+ dK, \tag{16}$$

where $H_s(\cdot)$ and $H_{SS}(\cdot)$ represent the first and second derivative of H with respect to S , and \bar{S} is the initial stock price. We denote

$$g_{i,t+\tau} = \frac{S_i(t, \tau) - S_i(t)}{S_i(t)} \tag{17}$$

as the return on the risky asset i with an investment horizon τ , where $S_i(t, \tau)$ is the price of the individual security at time $t + \tau$, and $S_i(t)$ is the price of the individual security at time t . In Proposition (4), we derive a model-free generalized measure of riskiness from option prices.

PROPOSITION 4. Let $R_{\delta,t}^Q[g_{i,t+\tau}]$ be the riskiness measure of the risky asset with simple return-payoff $g_{i,t+\tau} = (S_i(t, \tau) - S_i(t))/S_i(t)$. Let $C(S_i(t), K, \tau)$ be the price at time t of the call option with strike price K and maturity τ . Let $P(S_i(t), K, \tau)$ be the price at time t of the put option with strike price K and maturity τ . Denote $1 + r_{f,t+1}$ the return on the risk-free security, then $1 + R_{\delta,t}[g_{i,t+\tau}]$ is the fixed-point solution to (18):

$$\frac{r_f(t, \tau)}{1 + r_f(t, \tau)} \frac{1}{R_{\delta,t}^Q[g_{i,t+\tau}]} = \int_{S_i(t)}^\infty f_R[K]C(S_i(t), K, \tau) dK + \int_0^{S_i(t)} f_R[K]P(S_i(t), K, \tau) dK, \tag{18}$$

where

$$f_R[K] = \frac{(1 - \delta)}{(S_i(t)R_{\delta,t}^Q[g_{i,t+\tau}])^2} \left(1 + \frac{K/S_i(t) - 1}{R_{\delta,t}^Q[g_{i,t+\tau}]} \right)^{\delta-2}. \tag{19}$$

Equation (18) can be numerically solved to deduce critical wealth level $R_{\delta,t}^Q[g_{i,t+\tau}]$ (solve for the fixed-point $f[x] = x$).

PROOF. See Appendix B in the online appendix.² □

One of the important contributions of our paper is to provide a distribution-free generalized measure of $R_{\delta,t}^Q[g_{i,t+\tau}]$ that can be obtained from actively traded options and does not rely on any particular assumptions about the return distribution. Suppose an investor needs to find a one-month-ahead expected riskiness of a financial security. Under the physical measure, $R_{\delta,t}^Q[g_{i,t+\tau}]$ can only be obtained from the past historical data (e.g., daily returns over the past one year) and the investor has to use this historical measure as a proxy for future riskiness. However, this physical (or historical) measure does not reflect the market's expectation of future riskiness because history does not generally repeat itself. Using options' implied measures of $R_{\delta,t}^Q[g_{i,t+\tau}]$ solves this problem by making future riskiness observable, because option prices incorporate the market's expectation of future return distribution.

Another advantage of using options' implied measures of $R_{\delta,t}^Q[g_{i,t+\tau}]$ is that options with different maturities are highly liquid and traded in large volumes every day. In addition, the daily data on call and put options are publicly available and updated frequently. Hence, the option-based, forward-looking measures of $R_{\delta,t}^Q[g_{i,t+\tau}]$ can be estimated every day for a given investment horizon. For example, the $R_{\delta,t}^Q[g_{i,t+\tau}]$ measure obtained from a series of stock options with 30 days to maturity on day d gives one-month-ahead expected riskiness of the underlying stock from day d to $d + 30$.

5. Data and Variable Definitions

In this section, we first describe the equity options data used to estimate the generalized options' implied measures of riskiness. Then we explain the formation of risk-adjusted returns.

5.1. Equity Options Data

The daily data on call and put option prices and the corresponding strikes, maturities, and volatilities are from OptionMetrics. The OptionMetrics Volatility Surface computes the interpolated implied volatility surface separately for puts and calls using a kernel-smoothing algorithm that uses options with various strikes and maturities. The volatility surface

² We also derive, in Appendix C in the online appendix, the options' implied generalized measure of riskiness when the log return is used in place of the simple return. The results indicate that there is no significant empirical difference between the riskiness measures when simple or log returns are used.

data contain prices and implied volatilities for a list of standardized options for constant maturities and deltas. A standardized option is only included if there exist enough underlying option price data on that date to accurately compute an interpolated value. The interpolations are done each day, so no forward-looking information is used in computing the volatility surface. One advantage of using the Volatility Surface is that it avoids having to make potentially arbitrary decisions on which strikes or maturities to include when computing options' implied measures of riskiness for each stock. In our empirical analyses, we use out-of-the-money call and put option prices with expirations of 1, 3, 6, and 12 months to estimate the generalized forward-looking measures of riskiness. In Volatility Surface, at-the-money call (put) options have a delta of 0.50 (−0.50). Out-of-the-money call options have delta of 0.20–0.50, and out-of-the-money put options have delta of −0.20 to −0.50. The equity options data cover the sample from January 1996 to September 2008. At the end of each month t , Volatility Surface provides information for options with 30, 91, 182, and 365 days to maturity.³

5.2. Risk-Adjusted Returns of Individual Stocks

We obtain underlying stock return data from the Center for Research in Security Prices. The risk-adjusted return of an individual stock is defined as the stock's expected excess return per unit of risk.

Expected excess return. Expected return of an individual stock is measured by the average and holding period returns for the past 1, 3, 6, and 12 months. The three-month T-bill rate is used as a proxy for the risk-free interest rate. Expected return in excess of the risk-free rate gives the expected risk premium.

Risk. Risk of an individual stock i in month t is measured by the standard deviation of daily returns over the past 1, 3, 6, and 12 months (known as the realized volatility):

$$\sigma_{i,t} = \sqrt{\sum_{d=1}^{D_t} (R_{i,d} - \bar{R}_{i,d})^2}, \quad (20)$$

where $R_{i,d}$ is the return on stock i on day d , $\bar{R}_{i,d}$ is the average return on stock i on day d , and D_t is the number of days over the past 1, 3, 6, and 12 months.

Risk-adjusted return. Risk-adjusted return of an individual stock is defined as the ratio of expected excess return to risk. In our empirical analysis, alternative measures of risk-adjusted return are obtained from

³ Options that have an American-style exercise feature are priced using a proprietary pricing algorithm that is based on the industry-standard Cox–Ross–Rubinstein binomial tree model. This model can accommodate underlying securities with either discrete dividend payments or a continuous dividend yield.

the average and cumulative daily returns over the past 1, 3, 6, and 12 months, scaled by the realized volatility, which is also estimated with daily returns over the past 1, 3, 6, and 12 months.

6. Empirical Results

6.1. Riskiness Measure and Utility Function

Consider an investor with CRRA utility function

$$u(W) = \frac{W^\delta}{\delta}. \quad (21)$$

The representative investor with utility (21) will reject the gamble if, $E_t u(W_t + g_{t+1}) < u(W_t)$,

$$\frac{E_t(1 + g_{t+1}/W_t)^\delta - 1}{\delta} < 0. \quad (22)$$

Because the function $\phi^{(\delta)}[x]$ in Equation (9) is concave and our riskiness measure is the unique solution to $\phi^{(\delta)}[x] = 0$, inequality (22) implies that $1/(R_{1-\theta,t}[g_{t+1}]) < 1/W_t$. Therefore, the representative agent will reject the gamble g_{t+1} if her current wealth is below $R_{\delta,t}[g_{t+1}]$.

Merton's (1973) intertemporal capital asset pricing model indicates that the conditional expected excess return on a risky market portfolio is a linear function of its conditional variance, assuming that hedging demands are not too large:

$$E_t(R_{m,t+1}) = \theta \cdot E_t(\sigma_{m,t+1}^2), \quad (23)$$

where $E_t(R_{m,t+1})$ and $E_t(\sigma_{m,t+1}^2)$ are, respectively, the conditional mean and variance of excess returns on the market portfolio, and $\theta = 1 - \delta > 0$ is the relative risk aversion of market investors. Equation (23) establishes the dynamic relation that investors require a larger risk premium at times when the market is riskier. As presented in §1 of the online appendix, the risk-aversion parameter (θ) is estimated to be in the range of 3.05–3.17 and highly significant.

6.2. Relation to Aumann and Serrano (2008) and Foster and Hart (2009)

In this section, we show that our measure of riskiness is closely related to two recent pertinent measures of riskiness. The Foster and Hart (2009) operational measure of riskiness is a special case of our measure when $\delta = 0$. More importantly, when $\delta = 0$ and the riskiness measure is too large compared with the gain of the gamble g , our riskiness measure can be approximated by Foster and Hart (2009) and Aumann and Serrano (2008) riskiness measures. The Foster and Hart (2009) riskiness measure, $R^{\text{FH}}[g]$, is solution to

$$E \log \left(1 + \frac{g}{R^{\text{FH}}[g]} \right) = 0, \quad (24)$$

whereas the Aumann and Serrano (2008) riskiness measure, $R^{AS}[g]$, is solution to

$$E\left(1 - \exp\left(\frac{-g}{R^{AS}[g]}\right)\right) = 0. \quad (25)$$

Foster and Hart (2009) show that for every gamble g , there exists a unique real number $R^{FH}[g] > 0$ such that a simple strategy with critical-wealth function J guarantees no bankruptcy if and only if $J[g] \geq R^{FH}[g]$ for every gamble g . Thus, $R^{FH}[g]$ is the minimal wealth level at which g may be accepted. This minimal “reserve” $R^{FH}[g]$ needed for g holds for all decision makers without differentiating them in terms of their risk aversion. Aumann and Serrano’s (2008) measure of riskiness, $R^{AS}[g]$, is based on a duality axiom between riskiness and risk aversion and positive homogeneity of degree one.⁴ Aumann and Serrano (2008) define the riskiness of a gamble as the reciprocal of the constant absolute risk aversion of an individual who is indifferent between taking and not taking that gamble.

Aumann and Serrano’s (2008) measure, $R^{AS}[g_{t+\tau}]$, is an index of riskiness based on comparing the gambles in terms of their riskiness, whereas Foster and Hart’s (2009) measure, $R^{FH}[g_{t+\tau}]$, is defined for each gamble separately. That is, $R^{AS}[g_{t+\tau}]$ is based on risk-averse expected-utility decision makers, whereas $R^{FH}[g_{t+\tau}]$ does not require utility functions and risk aversion and just compares two situations: bankruptcy versus no bankruptcy or loss versus no loss. Hence, $R^{AS}[g_{t+\tau}]$ is based on the critical level of risk aversion, whereas $R^{FH}[g_{t+\tau}]$ is based on the critical level of wealth. The comparison between decision makers in Aumann and Serrano (2008)—being more or less risk averse—must hold at all wealth levels. In other words, $R^{AS}[g]$ looks for the critical risk-aversion coefficient regardless of wealth, whereas $R^{FH}[g]$ looks for the critical wealth regardless of risk aversion.

As shown in the earlier section, the relative risk aversion of market investors is estimated to be around three ($\theta = 3$), which implies $\delta = -2$ for the generalized measure of riskiness proposed in the paper. Because δ is economically and statistically different from zero, the empirical validity of the generalized measure nesting the original measures of riskiness is justified. Specifically, our estimates of θ suggest that an acceptance or rejection decision should be characterized by a utility function and a current wealth level; i.e., when measuring riskiness of individual stocks, one should look for the critical wealth by taking into account the investor’s risk tolerance. Hence, we define riskiness as the minimal wealth level at which g may be accepted,

but this minimal reserve needed for g is allowed to take different values, depending on the risk aversion of decision makers.

To illustrate the economic significance of the generalized measure of riskiness, we allow the decision maker to take any proportion of the offered gamble. She can overcome short-term losses by taking appropriately small proportions of the offered gambles depending on her risk tolerance. We use monthly data on the S&P 500 index options from January 1996 to September 2008 and compute the riskiness measure of Foster and Hart (2009) that gives the critical wealth regardless of risk aversion. Then, imposing $\delta = -2$, we estimate the generalized measure of riskiness using the same set of index options with 1, 3, 6, and 12 months to maturity.

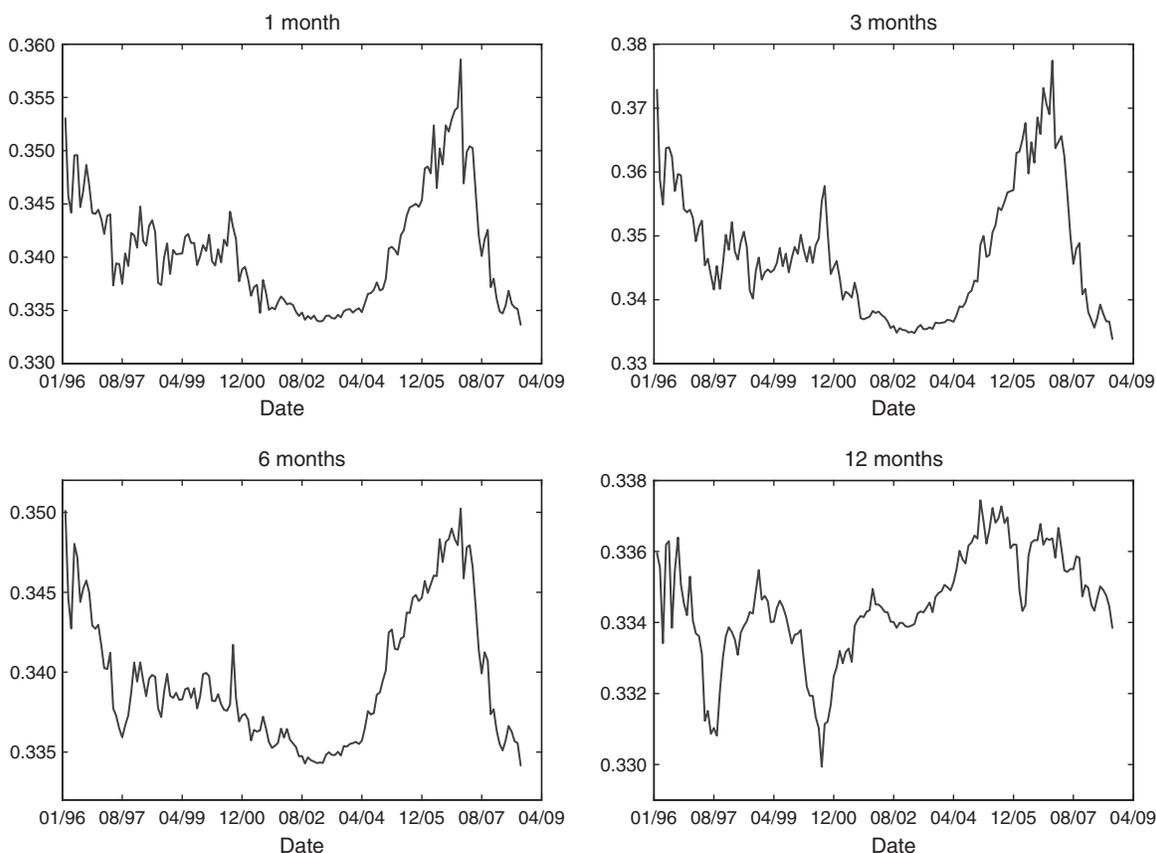
We examine the time-varying investment choice of a market investor with a relative risk aversion of three over the sample period of 1996–2008. Figure 1 displays the fraction of wealth that would be invested in the S&P 500 index by the market investor with $\theta = 3$ to avoid bankruptcy or extremely large losses.⁵ For investors with 1-, 3-, 6-, and 12-month investment horizons, the maximum weight she would give to the S&P 500 index varies from 33% to 38%, depending on the state of aggregate economy and investment horizon. Figure 1 demonstrates various levels of the investor’s risk-taking attitude, depending on the economic conditions. Because risk perception is higher during downturns of the economy, the risk-averse market investor reduces the allocation in the S&P 500 index especially during 2001 and between 2007 and 2008.

To gain further insight about the generalized measure of riskiness, Figure 2 presents the options’ implied measure of generalized riskiness with $\delta = -2$ and its nested version ($\delta = 0$). Note that the original measure of riskiness developed by Foster and Hart (2009) depends on the physical return distribution and corresponds to a special case of $\delta = 0$. However, to provide a consistent comparison in Figure 2, we generate the forward-looking measure of riskiness of Foster and Hart (2009) by imposing the constraint $\delta = 0$ in Equations (18) and (19) and using the S&P 500 index options with 1, 3, 6, and 12 months to maturity. Figure 2 shows that the generalized measure of riskiness plots above the forward-looking measure of Foster and Hart (2009) and the differences become

⁴Duality says that less risk-averse decision makers accept riskier gambles. Positive homogeneity represents the cardinal nature of riskiness, i.e., if g is a gamble, positive homogeneity implies that $2g$ is “twice as” risky as g , not just “more” risky.

⁵As discussed in Foster and Hart (2009), the decision maker’s wealth is $W_{t+1} = W_t + g_{t+1}$. However, in the generalized form, depending on the investor’s risk tolerance, the decision maker is allowed to take any proportion of the offered gamble; i.e., the decision maker’s wealth is $W_{t+1} = W_t + \alpha_t g_{t+1}$. Based on the properties of the riskiness measure in §3, the fraction of wealth invested in the gamble is the ratio of two riskiness measures with $\delta = 0$ and $\delta = -2$: $\alpha_t \leq R_{\delta=0,t}^Q[g_{t+1}]/R_{\delta=-2,t}^Q[g_{t+1}]$.

Figure 1 Risk-Taking Attitude of a Market Investor with $\theta = 3$



Notes. This figure presents the time-varying investment choice of a market investor with a relative risk aversion of three over the sample period of 1996–2008. For investment horizons of 1, 3, 6, and 12 months, the figure displays a fraction of the S&P 500 index that the market investor would buy to avoid bankruptcy or extremely large losses.

more significant when the level of riskiness is high and when the riskiness measures are estimated from the longer maturity options.

6.3. Asset Allocation Implications

Quantitative methods for asset allocation generally call for a constrained or unconstrained optimization. The constrained optimization yields optimal portfolio weights by maximizing expected return of a portfolio subject to a constraint on the portfolio’s risk, or minimizing the portfolio’s risk subject to a constraint on the portfolios’ expected return. The unconstrained optimization determines the optimum asset mix by maximizing the portfolio’s expected risk premium per unit of risk in a mean-variance framework or the expected value of a utility function approximated by the expected return and variance of the portfolio.

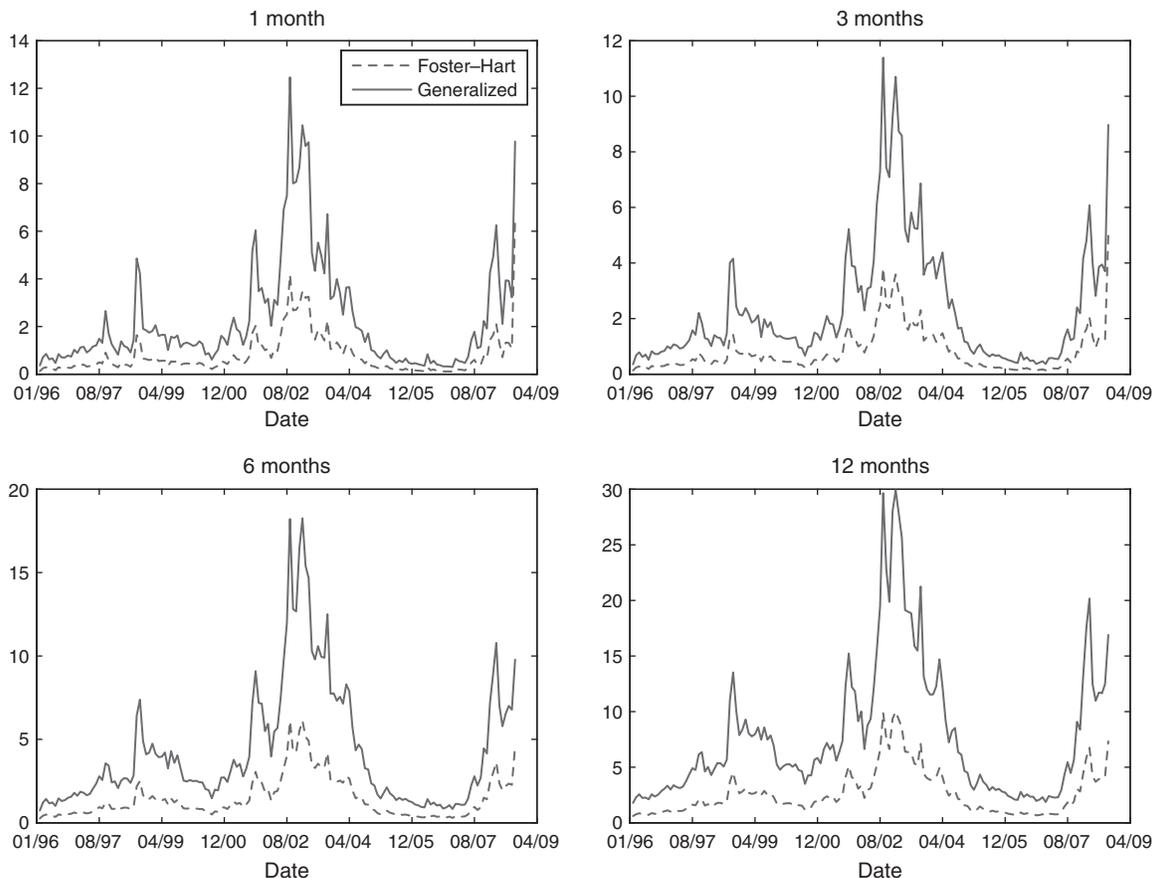
In all cases, the market risk of the portfolio is defined in terms of the variance or standard deviation of portfolio returns. Modeling portfolio risk with the traditional volatility measures implies that investors are concerned only about the average variation of individual asset returns, and they are not allowed to treat the negative and positive tails of the

return distribution separately. The standard risk measures determine the volatility of unexpected outcomes under normal market conditions. However, neither the variance nor the standard deviation can yield an accurate characterization of actual portfolio risk during extremely volatile periods. Therefore, the set of mean-variance efficient portfolios may produce an inefficient strategy for maximizing expected return of the portfolio while minimizing its risk.

One way to solve these issues is to replace variance with VaR and focus on the mean-VaR efficient portfolios.⁶ In this framework, investors are assumed to be loss averse; i.e., investors have in mind some disaster level of returns and they behave accordingly to minimize the probability of disaster. Hence, optimal portfolio in the mean-VaR framework can be selected

⁶ VaR measures market risk by determining how much the value of a portfolio (or a security) could decline over a given probability as a result of changes in market prices or rates. For example, if the given period of time is one day and the given probability is 1% (or the confidence level is 99%), the VaR measure would be an estimate of the decline in the portfolio value that could occur with a 1% probability over the next trading day.

Figure 2 Options' Implied Measures of Riskiness



Notes. This figure presents the generalized options' implied measure of riskiness with $\delta = -2$ and the options' implied riskiness measure of Foster and Hart (2009) with $\delta = 0$. The riskiness measures are obtained from the S&P 500 index options with 1, 3, 6, and 12 months to maturity for the sample period of January 1996–September 2008.

by maximizing expected return subject to a VaR constraint or by maximizing the expected value of a utility function approximated by the expected return and VaR of the portfolio.

As discussed, expected return and risk are two central elements in optimal portfolio choice. However, in practice, asset allocation also depends on the risk appetite of the investor. Risk tolerance is one of the most important factors influencing asset allocation because it takes into account an investor's willingness to take risks. A conservative or risk-averse investor would favor investments in which her capital is preserved, whereas an aggressive investor can risk losing her investment to generate higher profits. For example, a conservative investor would prefer a high ratio of fixed-income investments in the portfolio, whereas an aggressive investor would prefer to allocate more funds into stocks.

Optimal asset allocation based on the mean-variance and mean-VaR framework does not take into account the risk appetite of the investor. Diamond and Stiglitz (1974) indicate that whether an individual invests in a risky asset depends on how risky

the asset is and how averse the individual is to risk, which implies that increases in risk should affect more risk-averse individuals more than they do less risk-averse individuals. Therefore, appropriate definitions of increases in risk and in risk aversion should be closely linked. The riskiness measure introduced by Aumann and Serrano (2008) is aligned with this definition because, in their framework, individuals who are less risk averse are expected to invest in riskier assets.

Optimal portfolio selection may also depend on the wealth of the investor, because changes in the investor's wealth cause changes in the amounts and composition of the investor's consumption. Individuals generally spend more when one of two things is true: when they actually are richer (by objective measurement, for example, a bonus or a pay raise at work, which would be an income effect) or when they perceive themselves to be "richer" (e.g., the assessed value of their home increases, or a stock they own has gone up in price recently). Hence, the wealth effect is used to explain the increase in spending that results from an increase in perceived wealth. It is also

important to note that the investor’s risk appetite may be time varying and state dependent, and hence her risk-taking attitude may be influenced by changes in perceived wealth. Therefore, the initial wealth and expected future changes in wealth may affect the optimum asset mix. Foster and Hart (2009) propose a measure of riskiness that determines the critical wealth level; according to their model, investing in risky assets when the current wealth is below the critical wealth leads to bad outcomes, such as decreasing wealth and even bankruptcy in the long run.

In this paper, we introduce a generalized measure of riskiness that encompasses the original measures of Aumann and Serrano (2008) and Foster and Hart (2009). We show that an acceptance or rejection decision for risky gambles should be characterized by the risk tolerance and the current wealth of the decision maker. The newly proposed measure of riskiness computes the critical wealth by taking into account the risk aversion of investors. Hence, it provides a better characterization of riskiness of financial securities.

The fact that the generalized options’ implied measure of riskiness is sound and more comprehensive than the original measures does not mean that it would generate a more efficient strategy for maximizing risk-adjusted returns. To test the empirical performance of the newly proposed measure of riskiness, we first compute the options’ implied measure of riskiness for all stocks in OptionMetrics over the sample period of January 1996–September 2008. Then we check whether the forward-looking measures can predict the cross section of future risk-adjusted returns. We use the Fama and MacBeth (1973) cross-sectional regressions of one-month-ahead risk-adjusted returns of individual stocks on their generalized measure of riskiness:

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t} \cdot R_{\delta,t}^Q[g_{i,t+1}] + \varepsilon_{i,t+1}, \quad (26)$$

where $g_{i,t+1}$ is the average (or holding period) return on stock i in month $t + 1$; $r_{f,t+1}$ is the risk-free interest rate in month $t + 1$; $g_{i,t+1} - r_{f,t+1}$ is the expected excess return on stock i in month $t + 1$; $\sigma_{i,t+1}$ is the standard deviation of stock i in month $t + 1$; $(g_{i,t+1} - r_{f,t+1})/\sigma_{i,t+1}$ is the risk-adjusted return of stock i in month $t + 1$; $R_{\delta,t}^Q[g_{i,t+1}]$ is the generalized options’ implied Q measure of riskiness of stock i in month t obtained from individual equity options with 1, 3, 6, and 12 months to maturity; and $a_{0,t}$ and $a_{1,t}$ are the monthly intercepts and slope coefficients from the Fama–MacBeth regressions, respectively.

In our empirical analyses, $r_{f,t+1}$ is proxied by the three-month T-bill rate, and $g_{i,t+1}$ is estimated using the average and holding period returns for the past 1, 3, 6, and 12 months to be consistent with the definition of riskiness measure. Similarly, $\sigma_{i,t+1}$ is estimated using daily returns over the past 1, 3, 6, and

Table 1 Monthly Cross-Sectional Regressions of Risk-Adjusted Returns on the Generalized Options’ Implied Measure of Riskiness

Horizon	Average returns		Holding period returns	
	\bar{a}_0	\bar{a}_1	\bar{a}_0	\bar{a}_1
1 month	0.1123 (3.00) [0.13%]	−0.0131 (−3.41) [0.00%]	0.1123 (3.00) [0.13%]	−0.0131 (−3.41) [0.00%]
3 months	0.1157 (3.03) [0.15%]	−0.0049 (−3.21) [0.01%]	0.1850 (3.00) [0.00%]	−0.0201 (−2.62) [3.59%]
6 months	0.1210 (3.05) [0.14%]	−0.0181 (−2.82) [0.06%]	0.2571 (3.24) [0.00%]	−0.0335 (−2.29) [4.76%]
12 months	0.1267 (2.93) [0.13%]	−0.0215 (−2.21) [0.82%]	0.3565 (3.35) [0.00%]	−0.0049 (−1.86) [5.16%]

Notes. This table reports the average intercept and slope coefficients from the Fama and MacBeth (1973) cross-sectional regressions of one-month-ahead risk-adjusted returns of individual stocks on the stocks’ generalized options’ implied measure of riskiness:

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t} R_{\delta,t}^Q[g_{i,t+1}] + \varepsilon_{i,t+1},$$

where $g_{i,t+1}$ is the average (or holding period) return on stock i in month $t + 1$; $r_{f,t+1}$ is the risk-free interest rate in month $t + 1$; $g_{i,t+1} - r_{f,t+1}$ is the expected excess return on stock i in month $t + 1$; $\sigma_{i,t+1}$ is the standard deviation of stock i in month $t + 1$; $(g_{i,t+1} - r_{f,t+1})/\sigma_{i,t+1}$ is the risk-adjusted return of stock i in month $t + 1$; $R_{\delta,t}^Q[g_{i,t+1}]$ is the generalized options’ implied Q measure of riskiness of stock i in month t obtained from individual equity options with 1, 3, 6, and 12 months to maturity; and $a_{0,t}$ and $a_{1,t}$ are the monthly intercepts and slope coefficients from the Fama–MacBeth regressions. In the first stage, the generalized measure of riskiness is estimated using individual equity options with 1, 3, 6, and 12 months to maturity. The risk-adjusted returns of individual stocks are estimated using average and holding period returns along with the standard deviations from the past 1, 3, 6, and 12 months of daily data. In the second stage, the cross section of one-month-ahead risk-adjusted returns are regressed on the forward-looking measures of riskiness each month from January 1996 to September 2008. Newey and West (1987) t -statistics are reported in parentheses and Hodrick (1992) p -values are given in square brackets to determine the statistical significance of the average intercept and slope coefficients.

12 months, again to be consistent with the definition of riskiness measure.

Table 1 presents the time-series average intercepts and slope coefficients from Equation (26) over the sample period January 1996–September 2008. For all investment horizons, we obtain a negative and significant relation between the generalized measure of riskiness and the one-month-ahead risk-adjusted returns of individual stocks. Specifically, for the average return definition of $g_{i,t+1}$, the average slope on $R_{\delta,t}^Q[g_{i,t+1}]$ is estimated to be in the range of -0.0049 to -0.0215 with the Newey–West t -statistics ranging from -2.21 to -3.41 . For the holding period return definition of $g_{i,t+1}$, the average slope on $R_{\delta,t}^Q[g_{i,t+1}]$ is in the range of -0.0131 to -0.0490 , with the t -statistics ranging from -1.86 to -2.62 . These results indicate

Table 2 Risk-Adjusted Returns of Quintile Portfolios Formed Based on the Generalized Option's Implied Measure of Riskiness

	Average returns				Holding period returns			
	1 month	3 months	6 months	12 months	1 month	3 months	6 months	12 months
Low $R_{\delta,t}^Q[g_{i,t+1}]$	0.1033	0.1035	0.1019	0.1021	0.1033	0.1657	0.2130	0.2819
2	0.0884	0.0882	0.0901	0.0868	0.0884	0.1328	0.1674	0.2148
3	0.0560	0.0541	0.0552	0.0609	0.0560	0.0738	0.0874	0.1211
4	0.0085	0.0110	0.0138	0.0226	0.0085	0.0159	-0.0136	-0.0128
High $R_{\delta,t}^Q[g_{i,t+1}]$	-0.0492	-0.0492	-0.0430	-0.0218	-0.0492	-0.0814	-0.1147	-0.1192
High-low	-0.1525	-0.1527	-0.1449	-0.1239	-0.1525	-0.2471	-0.3277	-0.4011
	(-3.72)	(-3.72)	(-3.40)	(-2.98)	(-3.72)	(-3.71)	(-3.67)	(-3.90)

Notes. Quintile portfolios are formed every month from January 1996 to September 2008 by sorting individual stocks based on their options' implied generalized measure of riskiness. Quintile 1 (low $R_{\delta,t}^Q[g_{i,t+1}]$) is the portfolio of stocks with the lowest riskiness, and quintile 5 (high $R_{\delta,t}^Q[g_{i,t+1}]$) is the portfolio of stocks with the highest riskiness. The table reports the next month expected excess returns per unit of risk, where the expected return is measured by the average and holding period returns for the past 1, 3, 6, and 12 months and risk is measured by the standard deviation of daily returns over the past 1, 3, 6, and 12 months. The last row shows the differences in expected returns per unit of risk between high $R_{\delta,t}^Q[g_{i,t+1}]$ and low $R_{\delta,t}^Q[g_{i,t+1}]$ portfolios. Newey–West adjusted t -statistics are given in parentheses.

that the forward-looking options' implied measures of riskiness successfully predict the cross section of 1-, 3-, 6-, and 12-month-ahead risk-adjusted returns of individual stocks. Specifically, optionable stocks with higher riskiness have lower expected excess returns per unit of risk.⁷

Finally, we investigate whether the generalized measure of riskiness is able to rank stocks based on their expected returns per unit of risk. Quintile portfolios are formed every month from January 1996 to September 2008 by sorting individual stocks based on their options' implied measure of riskiness. As presented in Table 2, quintile 1 (low $R_{\delta,t}^Q[g_{i,t+1}]$) is the portfolio of stocks with the lowest riskiness, and quintile 5 (high $R_{\delta,t}^Q[g_{i,t+1}]$) is the portfolio of stocks with the highest riskiness. Table 2 reports the one-month-ahead expected excess returns per unit of risk of each portfolio. The expected return on individual stocks is measured by the average and holding period returns for the past 1, 3, 6, and 12 months and their risk is measured by the standard deviation of daily returns over the past 1, 3, 6, and 12 months as well, to be consistent with the definition of riskiness measure.

A notable point in Table 2 is that when moving from low $R_{\delta,t}^Q[g_{i,t+1}]$ to high $R_{\delta,t}^Q[g_{i,t+1}]$ portfolios, risk-adjusted returns of quintile portfolios decrease monotonically, indicating a strong, negative relation between the generalized measure of riskiness and the expected excess returns per unit of risk. This result holds for all investment horizons and for both definitions of expected return. Another notable point in Table 2 is that the differences in risk-adjusted returns

between the high $R_{\delta,t}^Q[g_{i,t+1}]$ and low $R_{\delta,t}^Q[g_{i,t+1}]$ portfolios are negative and highly significant without any exception. The last row of Table 2 shows that the average excess return per unit of risk of the stocks in the low $R_{\delta,t}^Q[g_{i,t+1}]$ portfolio is 0.12–0.15 per month higher than the average excess return per unit of risk of the stocks in the high $R_{\delta,t}^Q[g_{i,t+1}]$ portfolio. These risk-adjusted return differences are also highly significant with the Newey–West t -statistics ranging from -2.98 to -3.72. Somewhat stronger results are obtained for the holding period return definition of expected return.

Overall, the results from the Fama–MacBeth regressions and quintile portfolios provide corroborating evidence that the generalized measure of riskiness can be used to rank equity portfolios based on their expected returns per unit of risk and obtain a more efficient strategy for maximizing expected return of the portfolio while minimizing its risk.

6.4. Controlling for Skewness and Downside Risk

The original physical measure of riskiness developed by Foster and Hart (2009) resembles the statistical moments from the random variable characterizing the gamble, such as the expectation, volatility, skewness, and tails of a distribution. The riskiness measure of Foster and Hart (2009) condenses a physical probability distribution to a scalar one. Similarly, the generalized measure of riskiness introduced in this paper condenses the option's implied risk-neutral distribution into one number. It would be interesting to see how the generalized measure of riskiness compares to a risk-neutral measure of skewness and alternative measures of tail risk in predicting the cross-sectional variation in risk-adjusted returns.

Following Xing et al. (2010), we define the risk-neutral measure of skewness ($QSKEW$) as the difference between the out-of-the-money (OTM) put

⁷ Bakshi et al. (2011) find that forward variances inferred from option portfolios have predictive power for stock index returns. Consistent with their findings, our paper shows that options provide a good mechanism to amalgamate information in financial markets.

implied volatility and the average of the at-the-money (ATM) call and put implied volatilities:

$$QSKEW = OTM \text{ Put Vol} - \frac{1}{2}[(ATM \text{ Call Vol} + ATM \text{ Put Vol})]. \quad (27)$$

In addition to *QSKEW*, we use two more control variables, proxying for the left tail risk of optionable stocks. A primary tool for financial risk assessment is the value at risk (*VaR*), which is defined as the maximum loss expected on a portfolio of assets over a certain holding period at a given confidence level (probability). In our empirical analyses, *VaR* is estimated based on the lower tail of the empirical return distribution. For each month from January 1996 to September 2008, daily returns over the past one year are used to estimate the 1% and 5% *VaR* measures.⁸

VaR as a risk measure is criticized for not being subadditive; i.e., the risk of a portfolio can be larger than the sum of the stand-alone risks of its components when measured by *VaR*. Hence, managing risk by *VaR* may fail to incorporate diversification. Moreover, *VaR* does not take into account the severity of an incurred damage event. To alleviate these deficiencies, Artzner et al. (1999) introduce the expected shortfall (*ES*) measure of downside risk, which is defined as the conditional expectation of a loss given that the loss is beyond the *VaR* level. That is, the *ES* measure is defined as

$$ES_\alpha = E(X | X < VaR_\alpha), \quad (28)$$

where *X* represents the extreme return or loss, *VaR*_α is the value-at-risk threshold associated with the coverage probability α, and *ES*_α is the expected shortfall at the 100 × (1 − α) percent confidence level. As such, the expected shortfall considers loss beyond *VaR*. Equation (28) can be viewed as a mathematical transcription of the concept “average loss in the worst 100 × α percent cases.” For each month from January 1996 to September 2008, daily returns over the past one year are used to estimate the 1% and 5% *ES* measures.

We examine whether the significantly negative link between the generalized options’ implied measure of riskiness $R_{\delta,t}^Q[g_{i,t+1}]$ and future risk-adjusted returns remains intact after controlling for *QSKEW*, *VaR*, and *ES*. We run the following monthly Fama–MacBeth cross-sectional regressions:⁹

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t}R_{\delta,t}^Q[g_{i,t+1}] + a_{2,t}QSKEW_{i,t} + a_{3,t}VaR_{i,t} + \varepsilon_{i,t+1}, \quad (29)$$

⁸ We use two different measures of *VaR* to check its robustness as a predictive variable for the cross section of risk-adjusted returns.

⁹ Because *VaR* and *ES* are highly correlated in the cross section of individual stocks, we do not use them in the same regression, to avoid a multicollinearity problem.

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t}R_{\delta,t}^Q[g_{i,t+1}] + a_{2,t}QSKEW_{i,t} + a_{3,t}ES_{i,t} + \varepsilon_{i,t+1}. \quad (30)$$

As presented in Table 3, the average slope on *QSKEW* is negative and highly significant, with a few exceptions for the three-month average risk-adjusted returns. The average slopes on 1% *VaR* and 1% *ES* measures are also negative and significant, but the predictive power of 5% *VaR* and 5% *ES* measures are somewhat weaker, especially for the three-month average risk-adjusted returns. A notable point in Table 3 is that the average slope on $R_{\delta,t}^Q[g_{i,t+1}]$ remains negative and highly significant for all specifications of the cross-sectional regressions and for both the one-month and the three-month average risk-adjusted returns. Hence, controlling for the risk-neutral measure of skewness and the standard measures of downside risk does not affect our main findings.

7. Generalized Physical Measure of Riskiness

We introduce a generalized measure of riskiness, and the formal definition of the new measure falls under the physical distribution in §3, although the empirical applications have so far been provided from the options’ implied risk-neutral distribution. In this section, we extend our analysis to compare the riskiness measures under both physical (*P*) and risk-neutral (*Q*) distributions and examine the cross-sectional predictive power of both the riskiness and riskiness premium for optionable stocks.

As discussed, it is difficult to come up with accurate estimates under the physical measure, because generating empirical measures of riskiness requires precise estimates of the mean, standard deviation, and higher-order moments of the return distribution. It is well known that computing the moments of the return distribution is a difficult task, because one has to know the exact return distribution under the physical measure. Because this is not possible, one needs to make a distributional assumption, but then one needs a very long sample to generate reliable estimates of the moments under the assumed distribution. Instead of making a distributional assumption, we use the third-order Taylor series expansion to develop a generalized measure of riskiness from the physical measure.

We recall that the riskiness measure, $R_{\delta,t}^P[g_{i,t+\tau}]$, under the physical measure is defined by the following expression:

$$E_t(\mathcal{F}[g_{i,t+\tau}]) = 0, \quad (31)$$

with

$$\mathcal{F}[g_{i,t+\tau}] = \frac{(1 + g_{i,t+\tau}/R_{\delta,t}^P[g_{i,t+\tau}])^\delta - 1}{\delta}.$$

Table 3 Cross-Sectional Regressions of Risk-Adjusted Returns on $R_{\delta,t}^Q[g_{i,t+1}]$, Skewness, and Downside Risk

One-month average risk-adjusted return						Three-month average risk-adjusted return							
Intercept	$R_{\delta,t}^Q[g_{i,t+1}]$	<i>QSKEW</i>	1% <i>VaR</i>	5% <i>VaR</i>	1% <i>ES</i>	5% <i>ES</i>	Intercept	$R_{\delta,t}^Q[g_{i,t+1}]$	<i>QSKEW</i>	1% <i>VaR</i>	5% <i>VaR</i>	1% <i>ES</i>	5% <i>ES</i>
0.1232 (3.29) [0.07%]	-0.0141 (-3.53) [0.00%]	-0.1352 (-3.58) [0.00%]					0.1202 (3.18) [0.10%]	-0.0056 (-3.85) [0.00%]	-0.0145 (-1.18) [21.76%]				
0.1464 (3.37) [0.02%]	-0.0079 (-3.17) [0.02%]		-0.6419 (-2.28) [1.32%]				0.1398 (3.28) [0.04%]	-0.0087 (-2.58) [0.14%]		-0.5464 (-1.98) [1.82%]			
0.1596 (3.09) [0.03%]	-0.0060 (-2.64) [0.43%]			-1.5895 (-1.93) [2.45%]			0.1493 (3.02) [0.05%]	-0.0069 (-2.26) [0.63%]			-1.2409 (-1.52) [6.11%]		
0.1387 (3.37) [0.03%]	-0.0093 (-3.48) [0.00%]				-0.3978 (-2.58) [0.89%]		0.1347 (3.29) [0.04%]	-0.0103 (-2.86) [0.05%]				-0.3747 (-2.47) [0.79%]	
0.1535 (3.28) [0.02%]	-0.0066 (-2.84) [0.13%]					-0.9526 (-2.11) [1.86%]	0.1453 (3.19) [0.04%]	-0.0073 (-2.32) [0.38%]					-0.7884 (-1.77) [3.44%]
0.1544 (3.58) [0.01%]	-0.0075 (-2.91) [0.07%]	-0.1436 (-3.98) [0.00%]	-0.6724 (-2.36) [0.90%]				0.1462 (3.47) [0.02%]	-0.0100 (-3.33) [0.01%]	-0.0196 (-1.68) [8.33%]	-0.5711 (-2.07) [1.29%]			
0.1698 (3.30) [0.01%]	-0.0053 (-2.19) [1.47%]	-0.1555 (-4.40) [0.00%]		-1.6956 (-2.04) [1.61%]			0.1577 (3.21) [0.02%]	-0.0082 (-2.89) [0.09%]	-0.0233 (-2.03) [3.65%]		-1.3277 (-1.63) [4.44%]		
0.1459 (3.59) [0.01%]	-0.009 (-3.33) [0.01%]	-0.1396 (-3.94) [0.00%]			-0.4095 (-2.62) [0.66%]		0.1405 (3.48) [0.02%]	-0.0116 (-3.58) [0.00%]	-0.0195 (-1.67) [8.31%]			-0.3836 (-2.55) [0.59%]	
0.1621 (3.48) [0.01%]	-0.006 (-2.47) [0.47%]	-0.1471 (-4.17) [0.00%]				-1.0035 (-2.19) [1.27%]	0.1527 (3.39) [0.02%]	-0.0087 (-3.01) [0.03%]	-0.0215 (-1.87) [5.52%]				-0.8316 (-1.87) [2.48%]

Notes. This table reports the average intercept and slope coefficients from the Fama and MacBeth (1973) cross-sectional regressions of one-month-ahead risk-adjusted returns of individual stocks on the stocks' generalized options' implied measure of riskiness, risk-neutral measure of skewness (*QSKEW*), value-at-risk (*VaR*), and expected shortfall (*ES*):

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t}R_{\delta,t}^Q[g_{i,t+1}] + a_{2,t}QSKEW_{i,t} + a_{3,t}VaR_{i,t} + \varepsilon_{i,t+1}, \quad \frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t}R_{\delta,t}^Q[g_{i,t+1}] + a_{2,t}QSKEW_{i,t} + a_{3,t}ES_{i,t} + \varepsilon_{i,t+1},$$

where $g_{i,t+1}$ is the average (or holding period) return on stock i in month $t + 1$; $r_{f,t+1}$ is the risk-free interest rate in month $t + 1$; $g_{i,t+1} - r_{f,t+1}$ is the expected excess return on stock i in month $t + 1$; $\sigma_{i,t+1}$ is the standard deviation of stock i in month $t + 1$; $(g_{i,t+1} - r_{f,t+1})/\sigma_{i,t+1}$ is the risk-adjusted return of stock i in month $t + 1$; $R_{\delta,t}^Q[g_{i,t+1}]$ is the generalized options' implied Q measure of riskiness of stock i in month t obtained from individual equity options with one and three months to maturity. *QSKEW* is defined as the difference between the out-of-the-money put implied volatility and the average of the ATM call and put implied volatilities; 1% and 5% *VaR* and *ES* measures are obtained from daily returns over the past one year. $a_{0,t}$ is the monthly intercept; and $a_{1,t}$, $a_{2,t}$, and $a_{3,t}$ are slope coefficients from the Fama–MacBeth regressions. In the first stage, the generalized measure of riskiness is estimated using individual equity options with one and three months to maturity. The risk-adjusted returns of individual stocks are estimated using average and holding period returns along with the standard deviations from the past one and three months of daily data. In the second stage, the cross section of one-month-ahead risk-adjusted returns are regressed on $R_{\delta,t}^Q[g_{i,t+1}]$, *QSKEW*, *VaR*, and *ES* each month from January 1996 to September 2008. Newey and West (1987) t -statistics are reported in parentheses and Hodrick (1992) p -values are given in square brackets to determine the statistical significance of the average intercept and slope coefficients.

The Taylor expansion series of $\mathcal{T}[g_{i,t+\tau}]$ around $E_t(g_{i,t+\tau})$ allows us to express $\mathcal{T}[g_{i,t+\tau}]$ as

$$\begin{aligned} \mathcal{T}[g_{i,t+\tau}] &\approx \mathcal{T}[E_t g_{i,t+\tau}] + (g_{i,t+\tau} - E_t g_{i,t+\tau}) \mathcal{T}_g[E_t g_{i,t+\tau}] \\ &\quad + \frac{1}{2!} (g_{i,t+\tau} - E_t g_{i,t+\tau})^2 \mathcal{T}_{gg}[E_t g_{i,t+\tau}] \\ &\quad + \frac{1}{3!} (g_{i,t+\tau} - E_t g_{i,t+\tau})^3 \mathcal{T}_{ggg}[E_t g_{i,t+\tau}], \end{aligned} \quad (32)$$

with

$$\begin{aligned} \mathcal{T}_g[E_t g_{i,t+\tau}] &= \frac{1}{R_{\delta,t}^P[g_{i,t+\tau}]} \left(1 + \frac{E_t g_{i,t+\tau}}{R_{\delta,t}^P[g_{i,t+\tau}]} \right)^{\delta-1}, \\ \mathcal{T}_{gg}[E_t g_{i,t+\tau}] &= \frac{(\delta-1)(1 + E_t g_{i,t+\tau}/R_{\delta,t}^P[g_{i,t+\tau}])^{\delta-2}}{(R_{\delta,t}^P[g_{i,t+\tau}])^2}, \\ \mathcal{T}_{ggg}[E_t g_{i,t+\tau}] &= \frac{(\delta-1)(\delta-2)(1 + E_t g_{i,t+\tau}/R_{\delta,t}^P[g_{i,t+\tau}])^{\delta-3}}{(R_{\delta,t}^P[g_{i,t+\tau}])^3}. \end{aligned}$$

We place (32) into (31) and find that the riskiness measure is solution to the nonlinear equation

$$\begin{aligned} 0 &\approx \mathcal{T}[E_t g_{i,t+\tau}] + \frac{1}{2!} Var_t[g_{i,t+\tau}] \mathcal{T}_{gg}[E_t g_{i,t+\tau}] \\ &\quad + \frac{1}{3!} (Var(g_{i,t+\tau}))^{3/2} SKEW_t[g_{i,t+\tau}] \mathcal{T}_{ggg}[E_t g_{i,t+\tau}], \end{aligned} \quad (33)$$

where

$$\begin{aligned} Var_t[g_{i,t+\tau}] &= E_t(g_{i,t+\tau} - E_t g_{i,t+\tau})^2, \\ SKEW_t[g_{i,t+\tau}] &= \frac{E_t(g_{i,t+\tau} - E_t g_{i,t+\tau})^3}{(Var_t[g_{i,t+\tau}])^{3/2}}. \end{aligned}$$

To be consistent with the options' implied measure of riskiness, we first compute the mean, standard deviation, and skewness of daily returns over the past 1, 3,

6, and 12 months and then numerically back out the physical measure of generalized riskiness from Equation (33) with $\delta = -2$.¹⁰ To test the predictive power of the newly proposed physical measure of riskiness, we first use the Fama–MacBeth cross-sectional regressions:

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t}R_{\delta,t}^P[g_{i,t+\tau}] + \varepsilon_{i,t+1},$$

where $g_{i,t+1}$ is estimated using the average and holding period returns for the past 1, 3, 6, and 12 months. Similarly, $\sigma_{i,t+1}$ is estimated using daily returns over the past 1, 3, 6, and 12 months; $r_{f,t+1}$ is proxied by the three-month T-bill rate; $R_{\delta,t}^P[g_{i,t+\tau}]$ is the generalized physical measure of riskiness of stock i in month t ; and $a_{0,t}$ and $a_{1,t}$ are the monthly intercepts and slope coefficients from the Fama–MacBeth regressions, respectively.

Table 4 shows that the average slopes on the physical measure of riskiness are generally negative but statistically insignificant. This result holds for both the average and the holding period return definition of the Sharpe ratios, providing no evidence for a significant link between the physical measure of riskiness and the cross section of risk-adjusted returns.

We also investigate whether the generalized physical measure of riskiness is able to rank stocks based on their expected returns per unit of risk. Quintile portfolios are formed every month from January 1996 to September 2008 by sorting individual stocks based on their physical measure of riskiness. Confirming our earlier findings from the cross-sectional regressions, Table 5 shows that when moving from low $R_{\delta,t}^P[g_{i,t+\tau}]$ to high $R_{\delta,t}^P[g_{i,t+\tau}]$ portfolios, there is no monotonic pattern in risk-adjusted returns of quintile portfolios, indicating a flat relation between the physical measure of riskiness and the expected excess returns per unit of risk. Another notable point in Table 5 is that the spreads in risk-adjusted returns between low $R_{\delta,t}^P[g_{i,t+\tau}]$ and high $R_{\delta,t}^P[g_{i,t+\tau}]$ portfolios are statistically insignificant without any exception.

The physical measure of riskiness under P captures the actual risk, and the options' implied measure of riskiness under Q also incorporates the agent's preference for risk, and the difference between the two roughly has an interpretation as riskiness premium. We now test whether the riskiness premium is as informative as the riskiness itself when forecasting future risk-adjusted returns.¹¹

¹⁰ Figure 3 of the online appendix presents the generalized measure of riskiness (with $\delta = -2$) obtained from the physical distribution proxied by the mean, standard deviation, and skewness of daily returns over the past 1, 3, 6, and 12 months.

¹¹ Recent papers by Bollerslev et al. (2009) and Bali and Hovakimian (2009) provide evidence for a significant link between volatility risk

Table 4 Cross-Sectional Regressions of Risk-Adjusted Returns on the Generalized Physical Measure of Riskiness

Horizon	Average returns		Holding period return	
	\bar{a}_0	\bar{a}_1	\bar{a}_0	\bar{a}_1
1 month	0.0734 (2.16) [3.47%]	-0.2055 (-0.54) [64.95%]	0.0734 (2.16) [3.47%]	-0.2055 (-0.54) [64.95%]
3 months	0.0705 (2.09) [4.17%]	-0.0553 (-0.19) [84.99%]	0.1262 (2.40) [0.05%]	0.6395 (1.66) [5.34%]
6 months	0.0712 (2.12) [3.91%]	-0.1426 (-0.51) [51.93%]	0.1739 (2.63) [0.00%]	0.169 (0.71) [47.19%]
12 months	0.0786 (2.33) [2.28%]	0.0216 (0.14) [89.91%]	0.2594 (3.32) [0.00%]	-0.3530 (-1.79) [7.29%]

Notes. This table reports the average intercept and slope coefficients from the Fama and MacBeth (1973) cross-sectional regressions of one-month-ahead risk-adjusted returns of individual stocks on the stocks' generalized physical measure of riskiness.

$$\frac{g_{i,t+1} - r_{f,t+1}}{\sigma_{i,t+1}} = a_{0,t} + a_{1,t}R_{\delta,t}^P[g_{i,t+1}] + \varepsilon_{i,t+1},$$

where $g_{i,t+1}$ is the average (or holding period) return on stock i in month $t + 1$; $r_{f,t+1}$ is the risk-free interest rate in month $t + 1$; $g_{i,t+1} - r_{f,t+1}$ is the expected excess return on stock i in month $t + 1$; $\sigma_{i,t+1}$ is the standard deviation of stock i in month $t + 1$; and $(g_{i,t+1} - r_{f,t+1})/\sigma_{i,t+1}$ is the risk-adjusted return of stock i in month $t + 1$; $R_{\delta,t}^P[g_{i,t+1}]$ is the generalized physical P measure of riskiness of stock i in month t obtained from daily returns over the past 1, 3, 6, and 12 months; and $a_{0,t}$ and $a_{1,t}$ are the monthly intercepts and slope coefficients from the Fama–MacBeth regressions. In the first stage, the generalized measure of riskiness is estimated using Equation (33). The risk-adjusted returns of individual stocks are estimated using average and holding period returns along with the standard deviations from the past 1, 3, 6, and 12 months of daily data. In the second stage, the cross section of one-month-ahead risk-adjusted returns are regressed on the physical measures of riskiness each month from January 1996 to September 2008. Newey and West (1987) t -statistics are reported in parentheses and Hodrick (1992) p -values are given in square brackets to determine the statistical significance of the average intercept and slope coefficients.

We first generate the riskiness premium as the difference between the generalized options' implied and the generalized physical measures of riskiness, $R_{\delta,t}^Q[g_{i,t+\tau}] - R_{\delta,t}^P[g_{i,t+\tau}]$ and then form quintile portfolios every month from January 1996 to September 2008 by sorting individual stocks based on this spread. Table 6 shows that when moving from low $R_{\delta,t}^Q - R_{\delta,t}^P$ to high $R_{\delta,t}^Q - R_{\delta,t}^P$ portfolios, there is a significant decline in risk-adjusted returns of quintile portfolios, indicating a negative relation between the

premium and expected stock returns. Specifically, Bollerslev et al. (2009) find that stock market returns are predictable by the difference between model-free implied and realized variances, or the variance risk premium. In addition to this time-series evidence, Bali and Hovakimian (2009) present cross-sectional evidence and find a significant link between volatility risk premium and the cross section of expected returns at the individual stock level.

Table 5 Risk-Adjusted Returns of Quintile Portfolios Formed Based on the Generalized Physical Measure of Riskiness

	Average return				Holding period returns			
	1 month	3 months	6 months	12 months	1 month	3 months	6 months	12 months
Low $R_{\delta,t}^P [g_{i,t+1}]$	0.0371	0.0124	-0.0016	0.0047	0.0371	0.0339	0.0056	0.0690
2	0.0687	0.0502	0.0429	0.0693	0.0687	0.0796	0.0830	0.2102
3	0.0385	0.0617	0.0719	0.0929	0.0385	0.0734	0.1216	0.1472
4	0.0290	0.0377	0.0642	0.0627	0.0290	0.0556	0.0683	0.0397
High $R_{\delta,t}^P [g_{i,t+1}]$	0.0185	0.0120	0.0105	0.0122	0.0185	0.0214	0.0128	0.0216
High-low	-0.0186	-0.0004	0.0121	0.0075	-0.0186	-0.0125	0.0072	-0.0474
	(-1.30)	(-0.03)	(0.92)	(0.53)	(-1.30)	(-0.88)	(0.47)	(-1.60)

Notes. Quintile portfolios are formed every month from January 1996 to September 2008 by sorting individual stocks based on their generalized physical measure of riskiness defined in Equation (33). Quintile 1 (low $R_{\delta,t}^P [g_{i,t+1}]$) is the portfolio of stocks with the lowest riskiness, and quintile 5 (high $R_{\delta,t}^P [g_{i,t+1}]$) is the portfolio of stocks with the highest riskiness. The table reports the next month expected excess returns per unit of risk, where the expected return is measured by the average and holding period returns for the past 1, 3, 6, and 12 months and risk is measured by the standard deviation of daily returns over the past 1, 3, 6, and 12 months. The last row shows the differences in expected returns per unit of risk between high $R_{\delta,t}^P [g_{i,t+1}]$ and low $R_{\delta,t}^P [g_{i,t+1}]$ portfolios. Newey–West adjusted t -statistics are given in parentheses.

riskiness premium and the expected excess returns per unit of risk. Another notable point in Table 6 is that the differences in risk-adjusted returns between quintiles 5 and 1 are negative without any exception. The last row of Table 6 shows that these risk-adjusted return differences are also highly significant with the Newey–West t -statistics ranging from -3.35 to -3.76. Somewhat stronger results are obtained for the holding period return definition of expected return. Overall, we conclude that when predicting future returns, the riskiness premium is as informative as the riskiness itself.

Because the average daily returns over the past 1, 3, 6, and 12 months are negative for some periods in our sample (1996–2008), the physical riskiness measure can potentially be negative for some periods. As shown in Figure 3, the generalized measure of physical riskiness obtained from the S&P 500 index turns negative during a significant part of the sample

period. To generate an alternative measure of physical riskiness, instead of using the sample average return (i.e., the first moment of the empirical return distribution), we use a constant, positive expected rate of return on the S&P 500 index, 6% per annum (i.e., $\mu = 6\%$ in Equation (33)). Figure 4 of the online appendix plots this alternative measure of riskiness for the sample period January 1996–December 2008. Because we impose a positive expected market risk premium, physical riskiness becomes a function of standard deviation and skewness, and as shown in Figure 4, physical riskiness is now positive throughout the sample period.

To generate an alternative measure of physical riskiness for each stock in our sample, instead of using the average daily returns over the past 1, 3, 6, and 12 months, we use an expected return based on the capital asset pricing model

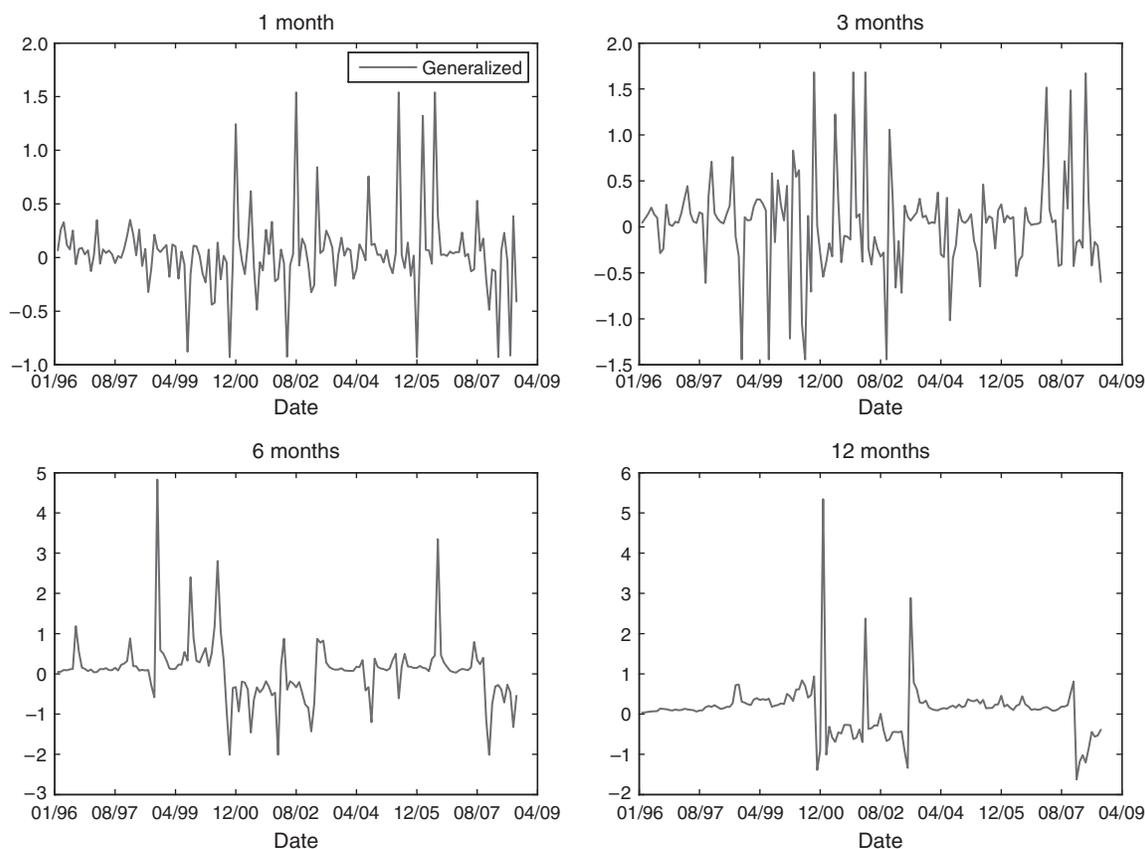
$$E(R_i) = r_f + \beta_i [E(R_m) - r_f], \tag{34}$$

Table 6 Risk-Adjusted Returns of Quintile Portfolios Formed Based on the Riskiness Premium

	Average return				Holding period returns			
	1 month	3 months	6 months	12 months	1 month	3 months	6 months	12 months
Low $R_{\delta,t}^Q - R_{\delta,t}^P$	0.0904	0.0920	0.0685	0.0572	0.0904	0.1260	0.1495	0.1542
2	0.0903	0.0935	0.0917	0.1043	0.0903	0.1315	0.1781	0.2375
3	0.0628	0.0523	0.0613	0.0783	0.0628	0.0824	0.1025	0.1609
4	0.0064	0.0162	0.0191	0.0354	0.0064	0.0129	-0.0059	-0.0110
High $R_{\delta,t}^Q - R_{\delta,t}^P$	-0.0423	-0.0556	-0.0495	-0.0276	-0.0423	-0.0783	-0.1078	-0.0639
High-low	-0.1327	-0.1476	-0.1180	-0.0848	-0.1327	-0.2043	-0.2573	-0.2181
	(-3.64)	(-3.71)	(-3.76)	(-3.35)	(-3.64)	(-3.76)	(-4.02)	(-3.23)

Notes. Quintile portfolios are formed every month from January 1996 to September 2008 by sorting individual stocks based on the riskiness premium, defined as the difference between the generalized option’s implied and the generalized physical measures of riskiness, $R_{\delta,t}^Q [g_{i,t+1}] - R_{\delta,t}^P [g_{i,t+1}]$. Quintile 1 (low $R_{\delta,t}^Q - R_{\delta,t}^P$) is the portfolio of stocks with the lowest riskiness premium, and quintile 5 (high $R_{\delta,t}^Q - R_{\delta,t}^P$) is the portfolio of stocks with the highest riskiness premium. The table reports the next month’s expected excess returns per unit of risk, where the expected return is measured by the average and holding period returns for the past 1, 3, 6, and 12 months and risk is measured by the standard deviation of daily returns over the past 1, 3, 6, and 12 months. The last row shows the differences in expected returns per unit of risk between high $R_{\delta,t}^Q - R_{\delta,t}^P$ and low $R_{\delta,t}^Q - R_{\delta,t}^P$ portfolios. Newey–West adjusted t -statistics are given in parentheses.

Figure 3 Generalized Physical Measure of Riskiness



Notes. This figure presents the generalized physical measure of riskiness with $\delta = -2$ for the sample period of January 1996–September 2008. The physical measures are obtained from daily returns on the S&P 500 index over the past 1, 3, 6, and 12 months.

where r_f is the risk-free interest rate (proxied by the three-month T-bill rate); $E(R_m)$ is the expected return on the market portfolio, which is assumed to be constant at 6% to be consistent with Figure 4; and β_i is the market beta of stock i , computed using daily returns on stock i and the market portfolio over the past 1, 3, 6, and 12 months. Once we estimate $E(R_i)$ for each stock i , we use it for μ in Equation (33).

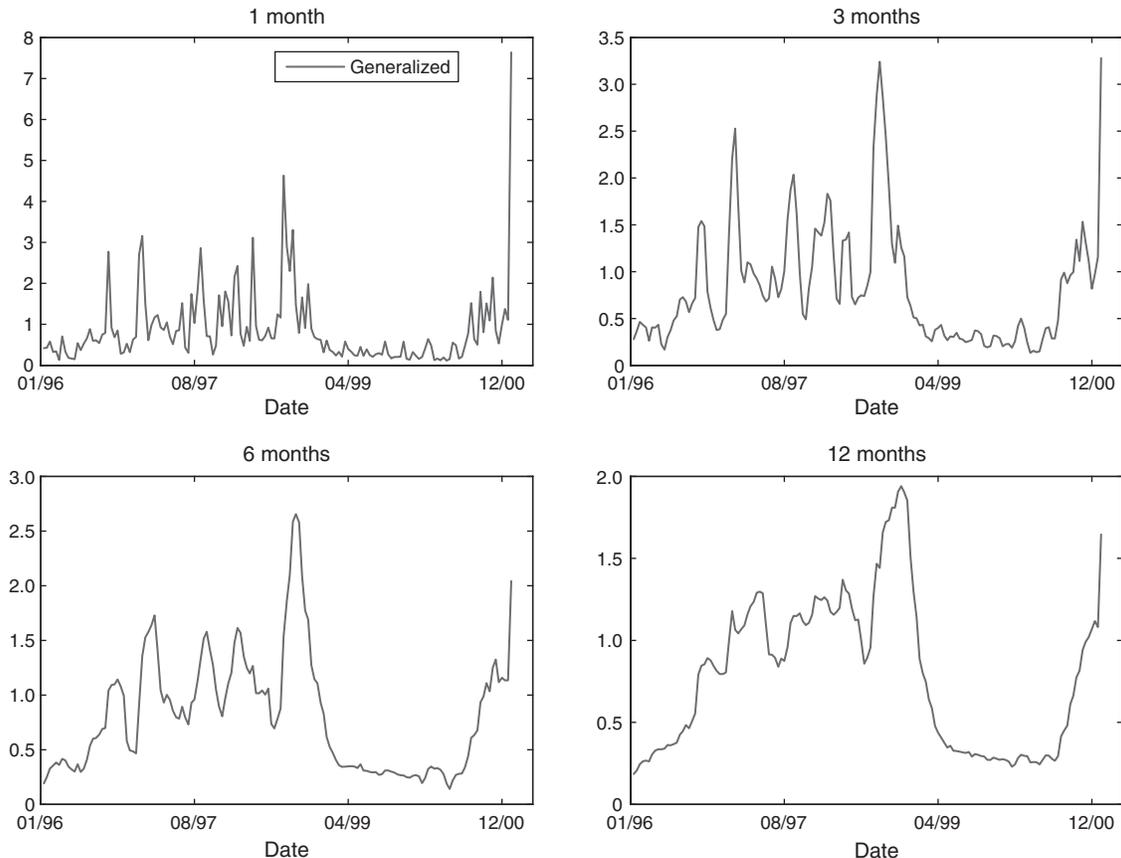
We test the predictive power of this alternative measure of physical riskiness using the Fama–MacBeth cross-sectional regressions. Table II of the online appendix shows that the average slopes on the alternative physical measure of riskiness are negative, but they are not statistically significant for all horizons. The results provide no evidence for a robust, significant link between the physical measure of riskiness and the cross section of risk-adjusted returns. Quintile portfolio results reported in Table III of the online appendix indicate a significantly negative link between physical riskiness and future risk-adjusted returns for 1-month and 6-month investment horizons, whereas the cross-sectional relation is weak for 3-month and 12-month horizons. Finally, we examine the predictive power of the riskiness premium,

defined based on the alternative measure of physical riskiness. Table IV of the online appendix shows that when moving from low $R_{\delta,t}^Q - R_{\delta,t}^P$ to high $R_{\delta,t}^Q - R_{\delta,t}^P$ portfolios, there is a significant decline in risk-adjusted returns of quintile portfolios, indicating a negative relation between the riskiness premium and the expected excess returns per unit of risk. Overall, the results in Table IV of the online appendix are very similar to our earlier findings in Table 6.

8. Conclusion

Risk measured by standard deviation, variance, and mean absolute deviation is crucial to optimal portfolio selection. Downside risk measured by semivariance, tail risk, and value at risk is essential to financial risk management. However, there are several issues with these standard and extreme measures of risk. First, standard deviation, variance, and mean absolute deviation determine only dispersion, taking little account of the asset’s actual values. They are not monotonic with respect to first-order stochastic dominance, i.e., a better risky asset with higher gains and lower losses may well have a higher standard deviation and thus be wrongly viewed as having a higher

Figure 4 Alternative Measures of Physical Riskiness



Notes. This figure presents alternative measures of physical riskiness with $\delta = -2$ for the sample period of January 1996–September 2008. The physical measures are obtained from daily returns on the S&P 500 index over the past 1, 3, 6, and 12 months and using $\mu_i = E(R_i)$ as defined in Equation (7) of the online appendix.

riskiness. The downside risk measures (e.g., VaR and expected shortfall) used extensively by banks and insurance companies depends on a parameter called a confidence level. The problem is that an appropriate value of the confidence level is not clear. Also, these downside risk measures ignore the gain side of the asset distribution, and even on the loss side, VaR concentrates only on that loss that hits the confidence level. The losses beyond the VaR threshold are not taken into account when computing the maximum likely loss of a portfolio. Because the standard measures of dispersion and downside risk do not satisfy the monotonicity and/or duality condition, the riskiness measures pioneered by Aumann and Serrano (2008) and Foster and Hart (2009) provide a better characterization of underlying risk than various measures already used by regulators and finance professionals. Moreover, the original measures of riskiness do not depend on any ad hoc parameters that need to be specified.

This paper contributes to the literature by proposing a generalized measure of riskiness that encompasses the existing measures of riskiness. Both

Aumann and Serrano (2008) and Foster and Hart (2009) introduce the riskiness measures based on the physical return distribution of gambles. Another contribution of this paper is that it launches a distribution-free measure of riskiness that can be obtained from actively traded options and does not rely on any particular assumptions about the return distribution. The paper shows that an acceptance or rejection decision should be characterized by a utility function and a current wealth level; i.e., when measuring riskiness of a gamble, one should look for the critical wealth by taking into account the investors' risk tolerance. Hence, riskiness is defined as the minimal wealth level at which the risky asset may be accepted, but this minimal reserve needed for the risky asset is allowed to take different values depending on the risk aversion of decision makers.

The paper also provides asset allocation implications of the generalized measure of riskiness and shows that the forward-looking measures predict the cross section of future risk-adjusted returns of individual stocks. Hence, the newly proposed measure of riskiness can be used to rank stock portfolios

based on their expected returns per unit of risk and can obtain a more profitable investment strategy. The significantly negative link between the generalized options' implied measure of riskiness and future risk-adjusted returns remains intact after controlling for the risk-neutral measure of skewness, value at risk, and expected shortfall.

Acknowledgments

The authors thank Wei Xiong (the department editor), an associate editor, and three anonymous referees for extremely helpful comments and suggestions. The authors also benefited from discussions with Gurdip Bakshi, Robert Engle, Armen Hovakimian, Liuren Wu, and seminar participants at Baruch College and the Graduate School and University Center of the City University of New York.

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