Optimal Generalized Location-Scale Distribution Investments

James N. Bodurtha, Jr.
McDonough School of Business
Georgetown University

and

Qi Shen
Stafford Partners

April 1994
latest revision –March 2004

Abstract

Generalized Location-Scale distributions include the normal, T-, and stable, as well as monotonic transformations of these distributions, such as the log. We consider investors choosing among either mutually exclusive returns or portfolio returns drawn from such distributions. The investors considered prefer more-to-less and may be either risk-averse or risk-seeking. For risk-seeking investors, the optimal set is an ordered subset of maximum-scale choices, and is a discrete function in mean-scale space. For risk-averse investors, the optimal sets are determined by the maximum mean for scale choices, and equal the well-known admissible sets. Fishburn’s Convex Stochastic Dominance (1974) is used to develop our results.

Please address correspondence to the first author: McDonough School of Business, Georgetown University, Old North 313, 37th & O Streets, NW, Washington, DC 20057, (202) 687-6351, fax: (202) 687-4031, and e-mail: bodurthj@georgetown.edu. We thank participants in the Georgetown Finance Workshop, Steve Brown, Phil Dybvig, Peter Fishburn, Keith Ord, Tierry Post and Shlomo Yitzhaki for helpful comments.
Optimal Generalized Location-Scale Distribution Investments

Abstract

Generalized Location-Scale distributions include the normal, T-, and stable, as well as monotonic transformations of these distributions, such as the log. We consider investors choosing among either mutually exclusive returns or portfolio returns drawn from such distributions. The investors that we consider prefer more-to-less and may be either risk-averse or risk-seeking. For risk-seeking investors, the optimal set is an ordered subset of maximum-scale choices, and is a discrete function in mean-scale space. For risk-averse investors, the optimal sets are determined by the maximum mean for scale choices, and equal the well-known admissible sets. Fishburn’s Convex Stochastic Dominance (1974) is used to develop our results.

Stochastic dominance and mean-scale (location-scale distribution) utility theories emphasize the reduction of an investment choice set to a smaller - more efficient - subset.\(^1\) When investors manifest both non-satiation and risk aversion, and return distributions are location-scale, mean-scale rules identify inefficient or dominated choice set elements. Any distribution with equal or lower mean and higher scale than another choice distribution will not be chosen [e.g. Bawa (1975).] The remaining undominated choices make up the associated admissible set.

For investors manifesting both non-satiation and risk-preference, Bawa (1976) has identified an admissible portfolio set among normally distributed investment choices. As he notes, his results extend to the location-scale distribution class.

Our question is whether or not generalized location-scale distribution admissible choice set elements will be chosen by investors manifesting non-satiation and either risk-aversion or risk-preference, the optimal set. In the context of both risk aversion and mutually exclusive investment decisions, we resolve an open issue in Meyers (1979),

\(^1\) See Levy (1992) for a thorough review of Stochastic Dominance.
Bawa-Goroff (1982) and Bawa et. al. (1985). We show that the most commonly considered generalized location-scale distribution admissible choices are optimal. For those location-scale distribution cases in which the distribution is maintained for portfolios of choices, we show that mean-scale admissible portfolio choices are optimal for risk-aversers. \(^2\) For risk-seekers, the admissible portfolio set is too large. Risk-seekers’ optimal set is equivalent for both mutually exclusive and portfolio investment choices.

Figure 1 depicts a multivariate normal distribution example. Analyzing the data set considered in Post (2003): the Fama-French Size/Market-to-Book portfolios for July 1963 – October 2001, Figure 1 – Panel A: depicts the optimal set for mutually exclusive investments, and Figure 1 – Panel B: depicts the optimal set for portfolio investments.

The salient features of the plots are the lack of convexity and discreteness. Only the risk-aversers’ portfolio choice optimal set has a convex surface. The risk-aversers’ mutually exclusive investment choice set is a discrete step function. This function rises from the choice with lowest mean and scale to the choice with highest mean and scale. Dominated alternatives have means that are less than an optimal set choice and higher scale.

\(^2\) For exchangeable distributions, such as the normal, \(T\) and stable, admissible portfolio choices or mutually exclusive investment choices are optimal. For the generalized cases, including the log transforms of these distributions, such as log-normal, the mutually exclusive admissible choices are optimal. However, portfolio returns drawn from these generalized distributions are not location scale. In these cases, the two parameter mapping between admissible and optimal choices is not preserved. For a general discrete distribution class and risk-averse investors, Yitzhaki and Mayshar (1997) and Post (2003) have shown that admissible portfolios are optimal.
Figure 1

Panel A: Mutually Exclusive Investment Mean-Variance Optimal Set
(Fama-French 25 Book-to-Market and Size and CRSP Market Portfolios
monthly returns with no short sales: July 1963 - October 2001)

Panel B: Portfolio Investment Mean-Variance Optimal Set
(Fama-French 25 Book-to-Market and Size and CRSP Market Portfolios
monthly returns with no short sales: July 1963 - October 2001)
The paper is organized as follows. First, we provide notation and state definitions. Our second section treats risk-averse investor optimal mutually exclusive choice in a first subsection and their portfolio problem in a second subsection. The third section states necessary and sufficient optimality conditions for investor with risk preference, and a linear programming–based algorithm is specified for the allocation problem. Our main result draws on previous sections to identify the location-scale distribution optimal set for either risk-averse or risk-seeking investors. In Section Five, our remarks conclude. The Lemmas that support our Section 2 and 3 proofs are stated and proven in an appendix.

1. Notation and Definitions

Generally accepted observable behavior has led to the following classes of continuously differentiable utility functions, \( u(\cdot) \):

(i) nonsatiation axiom: \( u' > 0 \)

(ii) risk aversion: \( u' > 0, u'' < 0 \), or
risk preference: \( u' > 0, u'' < 0 \)

Adopting the notation of Bawa (1975), let the uncertain prospects be characterized by random variables \( x_{i}, i=1,2,\ldots,n+1 \), with known continuous probability distribution functions defined over an open interval \( R_{1} \) given by \( (a, b) \), \( a < b \). For the location-scale distributions of interest, the interval is either the positive half or all of the real line.

Let the following progressively restrictive set of utility functions, \( u(\cdot) \), describe the decision maker's preferences. The utility functions are defined over the space \( R_{1} \) of realizations of a random variable \( x \):

\[
U_{1} = \left\{ u(x) \mid u(x) \text{ is finite, } u'(x) > 0, \text{ for all } x \in R_{1} \right\},
\]
We also consider two classes of distributions:

**Location-Scale:** \( F(\mu, \sigma, x) \in \Psi_0 \), if \( F(\mu, \sigma, x) = \psi \left( \frac{x - \mu}{\sigma} \right) \) for all \( x \in [-\infty, \infty] \) and \( \sigma > 0 \). For a \( \Psi_0 \) distribution, the expected value, \( E_iX \), equals \( \mu_i + \sigma_i A \), independent of location and scale.

**Generalized Location-Scale:** \( F(\mu, \sigma, x) \in \Psi_1 \), if \( F(\mu, \sigma, x) = \psi \left( \frac{\phi(x) - \mu_i}{\sigma_i} \right) \) \( \phi'(t) > 0 \) for all \( t \in [-\infty, \infty] \) and \( \sigma_i > 0 \).

Examples of \( \Psi_0 \) distributions include the normal, T and Cauchy. \( \Psi_1 \) distributions include the log transforms of the \( \Psi_0 \) distributions.\(^3\)

### 2. Risk-Averse Behavior

Under the risk aversion assumption, these definitions lead to the following well-known second-order stochastic dominance theorem and definition:

**Theorem 1:** Second-Order Stochastic Dominance (SSD). For any two cumulative location \((\mu)\), scale \((\sigma)\) and transform, \(\psi(\mu, \sigma, \phi(t))\) and \(\phi(t)\), distributions \(F(\mu_i, \sigma_i, t)\) and \(F(\mu_j, \sigma_j, t)\) with \( t \in [-\infty, \infty] \), distribution \(i\) is (strictly) preferred to distribution \(j\) for all utility functions in \(U_2\), if and only if

\[
\int_a^x F(\mu, \sigma, t) dt \leq \int_a^x F(\mu_i, \sigma_i, t) dt \quad \forall x \in \mathbb{R} \quad \text{and} \quad \text{for some} \ x \in \mathbb{R}.
\]

or \( E_iX \geq E_jX \quad (E_iX > E_jX) \) and \( \sigma_i < \sigma_j \), Bawa (1975).

**Definition:** SSD Admissible Set

---

\(^3\) The Stable, F, LaPlace, Extreme Value, and Logistic Distributions are other \( \Psi_0 \) distributions defined over the real line. Other location-scale distributions over bounded domains are the Gamma, Beta, Uniform, Triangular, and Weibull distributions. (The most common transformation for \( \Psi_1 \) distributions is \( \phi(t) = (t^\lambda - 1)/\lambda \), and \( \phi(t) = \log(t) \) for \( \lambda = 0 \).) Bawa (1979)
A subset A of choice set P, its members are not second-order stochastically dominated.

If a choice in P is not in subset A (not admissible), then all investors unanimously drop it from consideration. By dropping these choices, the SSD admissible set substantially reduces the full choice set.

In the case of choices from location-scale distributions, convex second-order dominance is an optimal choice rule.

**Definition:** Convex Second-Order Stochastic Dominance (CSSD): A distribution function \( F_{n+1} \) is convex second-order stochastically dominated by \( \{F_i, i=1,2,\ldots,n\} \), if \( \forall u \in U_2 \), there exists an \( F_j \in \{F_1,F_2,\ldots,F_n\} \) such that

\[
\int_a^b U(x) dF_j(x) \geq \int_a^b U(x) dF_{n+1}(x). \quad (2)
\]

Correspondingly, we introduce the CSSD optimal set.

**Definition:** CSSD Optimal Set

A subset C of choice set A is CSSD optimal if \( \forall a \in C \), a choice is not CSSD dominated by any other members of A (or P).

Since CSSD optimality is more restrictive than the usual SSD admissibility, the CSSD optimal set is generally smaller than the SSD admissible set.

Let \( \lambda=(\lambda_1,\lambda_2,\ldots,\lambda_n) \), \( \lambda \in \Lambda_n \) with \( \lambda_i \geq 0, i=1,2,\ldots,n \) and \( \sum_{i=1}^n \lambda_i = 1 \). Following Fishburn (1974), we state the convex generalization of Theorem 1.

**Theorem 2:** Convex Second-Order Stochastic Dominance (CSSD). \( F_{n+1} \) is convex second-order stochastically dominated by \( \{F_i, i=1,2,\ldots,n\} \), iff \( \exists \lambda \in \Lambda_n \) such that

\[
\sum_{i=1}^n \lambda_i \int_a^x F_i(t) dt \leq \int_a^x F_{n+1}(t) dt \quad \forall x \in R \text{ (and < for some } x \in R) \quad (3)
\]
Conversely, if \( F_{n+1} \) is not convex second-order stochastically dominated by \( \{ F_i, i=1,2,\cdots,n \} \), then it is optimal:

\[
\forall \lambda \in \Lambda_n, \exists x \in R, \int_a^x F_{n+1}(t) \, dt < \sum_{i=1}^n \lambda_i \int_a^x F_i(t) \, dt \iff \exists u \in U_2, u(F_i) < u(F_{n+1}), \forall \{i=1,2,\cdots,n\}
\]

(4)

There has not been specific identification of location-scale distribution optimality.

We also define another important concept relevant to investment choice, the efficiency of a choice set.

**Definition:** Second-Order Efficient Set

A subset \( E \) of choice set \( P \) is second-order efficient if it contains the maximizers for all \( U_2 \).

Obviously, investors with non-satiation and risk-aversion attributes should only evaluate the minimal second-order efficient choice set in order to make their investment decisions. The minimal efficient choice set is the CSSD optimal set.

### 2.1 CSSD Optimal Mutually Exclusive Investments

Convex Stochastic Dominance (CSD) identifies choice distribution mixtures that dominate other elements of the choice set (the dominated elements). Fishburn (1974) shows that any choice dominated by a mixture of other alternatives will not be chosen. Conversely, any choice that is not so dominated will be chosen and is in the optimal set.

Though a bit tedious, our method of proof is straightforward. For generalized location-scale distributions, the appropriate SD decision rule is second-order (SSD). Since these distributions cross except in the case of equal scales, first-order stochastic dominance (FSD) is precluded. Under Convex Second-Order Stochastic Dominance

---

4 Bawa and Goroff (1982) demonstrate the equivalence between SD admissibility and efficiency for the portfolio allocation problem.
(CSSD), we show that the set of mixture distributions necessary to dominate any member of the admissible set is empty. Hence, the admissible set is optimal.

For mutually exclusive choices, the choice space may be written as the following:

$$P = \left\{ \sum_{i=1}^{n} \lambda_i F_i : \lambda_i \in \Lambda_n, F_i \text{ is location-scale for } i=1,2,\ldots,n \right\}$$  \hspace{1cm} (5)

The set of non-SSD dominated distributions (the admissible set) is no smaller than the set of non-CSSD dominated distributions (the optimal set). However, the following theorem shows that in the case of generalized location-scale distributions, these two concepts coincide. In this case, the two choice sets are identical.

**Proposition 1:**

Given a set of general location-scale distributions \( \Phi = \{F_1,F_2,\ldots,F_n,F_{n+1}\} \), if \( \Phi \) is a \( U_2 \) admissible set, then it is also the CSSD optimal set.\(^5\)

**Proof:** Our proof requires two Lemmas. We state and prove these Lemmas in our appendix.

Since \( \Phi \) is an admissible set, the distributions are mutually undominated. In the location-scale distribution case, SSD is equivalent to the mean-scale decision rule, and we can order the distributions in \( \Phi \) in such a way that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_n, \quad \text{and} \quad \mu_1 < \mu_2 < \cdots < \mu_n.$$

The mean and scale of distribution \( F_{n+1} \) may be anywhere in the sequence of \( F_1,F_2,\ldots,F_n \).

---

\(^5\) In the context of previous Generalized Location-Scale distribution family-related footnotes, mean-scale admissible T distributions with the same degree of freedom, stable distributions with the same characteristic exponent and skewness and log-normal distributions are optimal.
Case 1: \( \sigma_{n+1} < \sigma_n = \max_{1 \leq j \leq n} \{ \sigma_j \} \). We divide the set \( \Phi \) in two parts: \( \Phi_1 = \{ F_1, \cdots, F_k \} \), and \( \Phi_2 = \{ F_{k+1}, \cdots, F_n \} \), such that \( \mu_k < \mu_{n+1} < \mu_{k+1} \) and \( \sigma_k < \sigma_{n+1} < \sigma_{k+1} \).

We can take a degenerate distribution as a special case of the distribution, by defining its scale to be zero. We replace the set \( \Phi_1 \) with another set \( \hat{\Phi}_1 \) such that
\[
\mu_\hat{1} = \mu(F_i), \quad \sigma_\hat{1} = 0, \quad i = 1, 2, \cdots, k.
\]

If \( F_{n+1} \) cannot be dominated by \( \hat{\Phi}_1 \cup \Phi_2 \), then \( F_{n+1} \) also can't be dominated by \( \phi_{n+1} \cup \Phi_2 \) (since each member of \( \Phi_1 \) is dominated by the corresponding member in \( \hat{\Phi}_1 \)).

For members of set \( \Phi_2 \), we choose a sufficiently small number, \( r \), such that the Scale Dominance Rule can be applied to each element of \( \Phi_2 \). For simplicity, we keep the notation of \( F_i, i = 1, \cdots, k \), instead of \( \hat{F}_i \).

From Lemma 1, for any given \( \lambda_j > 0 \), there exists an \( r_j \) such that
\[
\int_{-\infty}^{r_j} F_{n+1}(t)dt < \int_{-\infty}^{r_j} F_j(t)dt, \quad j = k + 1, \cdots, n. \tag{6}
\]
Therefore, there exists a real number \( r \in \mathbb{R}, \quad r < \min \{ \mu_i; i = 1, \cdots, k, r_j; j = k+1, \cdots, n \} \), for any given \( \lambda \in \Lambda_n \),
\[
\int_{-\infty}^{r} F_{n+1}(t)dt < \sum_{j=k+1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt = \sum_{j=k+1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt + \sum_{j=1}^{k} \lambda_j \int_{-\infty}^{r} F_j(t)dt \tag{7}
\]
\[
\int_{-\infty}^{r} F_{n+1}(t)dt < \sum_{j=1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt
\]
Here, we have used the fact that \( \int_{-\infty}^{r} F_j(t)dt = 0 \) for \( j = 1, \cdots, k \), since \( r < \mu_j \).

We have shown that \( F_{n+1} \) is not CSSD dominated by \( \{ F_1, \cdots, F_n \} \).
Case 2: $\sigma_{n+1} > \max_{1 \leq j \leq n} \left\{ \sigma_j \right\} = \sigma_n$

In this case, from Lemma 1, there exists a sufficiently larger number $r_j$, such that

$$\int_{r_j}^{+\infty} F_j(t) dt < \int_{r_j}^{+\infty} F_{n+1}(t) dt \quad j = 1, 2, \ldots, n.$$  \hspace{1cm} (8)

Thus,

$$\int_{\infty}^{r} F_{n+1}(t) dt = 1 - \int_{r}^{+\infty} F_{n+1}(t) dt < 1 - \sum_{j=1}^{n} \lambda_j \int_{r}^{+\infty} F_j(t) dt = \sum_{j=1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t) dt$$  \hspace{1cm} (9)

where $r > \max_{1 \leq j \leq n} \left\{ r_j \right\}$. In this case, we have shown that $F_{n+1}$ can't be CSSD dominated by $\Phi$.

Q.E.D.

For sets of investors with non-satiation and risk-aversion attributes, $U_2$, who face mutually exclusive location-scale distribution investment returns, we have shown that the second-order stochastic dominance (SSD) admissible set is the Bawa et. al. (1985) optimal set and the Bawa-Goroff (1982) strictly-best set. We now consider risk-averse investors’ portfolio choices.

2.2 CSSD Optimal Portfolio Choices

For risk-averse investor portfolio choices, Baron (1977) has shown that a choice vector, $\pi$, dominates the associated mixed strategy, $\lambda \pi$, for all strictly concave von Neumann-Morgenstern utility functions. Yitzhaki-Mayshar (1997) proved this result in the context of Marginal Stochastic Dominance for general discrete distributions and the normal distribution. In proving this result for the location-scale distribution case, we use
two Lemmas from Appendix: Our CSSD $\Psi_0$ distribution efficient portfolio proposition follows: 

**Proposition 2:** The $\Psi_0$ location-scale distribution mean-scale efficient frontier choice is the CSSD optimal set.

**Proof:** Given Lemma 4, any mixture of alternatives is dominated by an associated portfolio. Any portfolio not associated with the mean-scale efficient frontier is dominated by some element of the set of portfolios on the efficient frontier. Therefore, mean-scale efficient portfolio choices dominate mixtures of portfolio distributions, and mean-scale efficient portfolios are CSSD optimal.

Like Proposition 1 for the CSSD mutually exclusive choice admissible and optimal sets, Proposition 2 shows that the entire mean-scale admissible and efficient portfolio frontier is optimal. Nevertheless, this result only holds for $\Psi_0$ distributions for which portfolios remain in the same location-scale distribution class.

### 3. Risk-Seeking Behavior

Building on the seminal work of Friedman-Savage (1948) and Kahneman and Tversky (1979), a number of authors have evaluated both global and local risk-seeking behavior, $U^2$.

Global optimality for risk-seeking investors has not been addressed for either mutually exclusive or portfolio investments. An approach following the Bawa et. al. 1985 algorithm may be used to solve both problems. The required changes are two:

---

6 Portfolios formed from $\Psi_1$ distributed random return choices do not stay in the location-scale distribution class. The location-scale family of distributions of Bawa (1975), $\Psi_0$, are among the distributions that are exchangeable up to a linear transformation. Schectman and Yitzhaki (1999) have shown that the Gini risk measure has portfolio Gini properties analogous to the covariance. For the distributions of concern, Mean-Gini are sufficient statistics for risk, and our Proposition 2 may equivalently be stated in these terms. Thus, the Yitzhaki-Mayshar (1997) optimality result for Normal Distributions should extend to the location-scale distributions that we consider.

7 For examples, see Barberis, Huang and Santos (2001), Hartley-Farrell (2001), and Post-Levy (2003).
First, the second order stochastic dominance integral inequality of Theorem 2 (equation (3)) switches from a less than equal to inequality to a greater than equal to inequality. Second, the non-satiation condition on expected returns must be explicitly constrained.

**Theorem 3:** Risk-Seeking Convex Second-Order Stochastic Dominance (RS-CSSD).

\( F_{n+1} \) is convex second-order stochastically dominated for a risk seeking investor by \( \{F_i, i=1,2,\ldots,n\} \), iff \( \exists \lambda \in \Lambda_n \) such that

\[
\sum_{i=1}^{n} \lambda_i \int_{a}^{x} F_i(t) \, dt \geq \int_{a}^{x} F_{n+1}(t) \, dt \quad \forall x \in \mathbb{R} \quad \text{(and < for some } x \in \mathbb{R})
\]

(10)

\[
\sum_{i=1}^{n} \lambda_i \mu_i > \mu_{n+1}
\]

A discrete approximation to the analogous risk-averse investor CSSD inequality set was implemented by BBRS for the mutually exclusive investment problem. This approach was insufficient for determining optimality. As we have shown, the lower tails of location-scale distribution choices completely dominate the risk-averse decision analysis.

However, the risk-seeking utility problem analysis isn’t dominated by tail behavior. The lower tail is dominated absolutely by the necessary higher variance choice and the upper tail is dominated by the integrated distributions. Therefore, a discrete approximation will converge to the continuous problem value for the risk-averse investor optimal choice problem, (10).\(^8\)

---

\(^8\) Since location-scale densities are analytic and smooth, the BBRS Third-Order Stochastic Dominance e-dominance semi-infinite programming application may be adjusted to identify the optimal and dominated sets. For our application, their mixed set will be null.
For the discrete approximation of location-scale distribution $i$, we define

$$S_{it} = \sum_{\tau=1}^{t} F\left(\mu_{i}, \sigma_{i}, \tau \right), \text{ sum of the discrete approximation to cumulative distribution}$$

$$F_{it} = \sum_{\tau=1}^{t} f\left(\mu_{i}, \sigma_{i}, \tau \right), \text{ a discrete approximation to cumulative distribution}$$

For $N+1$ distributions of which distribution $N+1$st is under test for risk-seeking dominance, the BBRS risk-averse linear programming specification analogue is the following:9

$$\begin{align*}
\begin{bmatrix}
S_{11} & \cdots & S_{Nh} \\
\vdots & \ddots & \vdots \\
S_{1h} & \cdots & S_{Nh}
\end{bmatrix}
\begin{bmatrix}
\lambda
\end{bmatrix}
\geq
\begin{bmatrix}
S_{N+11} \\
\vdots \\
S_{N+1h}
\end{bmatrix},
\end{align*}$$

\[
\left(\mu - \mu_{N+1} e\right)' \lambda \geq 0,
\]

$\lambda$ is the $N \times 1$ solution vector
$e$ is an $N \times 1$ unit vector
$\mu$ is the $N \times 1$ potentially dominating distributions' mean vector
$\mu_{N+1}$ is the mean of the potentially dominated distribution

A unit solution to this linear program is both necessary and sufficient for dominance. In this case, one of the location-scale distributions with non-zero weights will be at least as preferred as the dominated alternative for all investors manifesting non-satiation and risk-preference. We have our third proposition:

---

9 The BBRS risk-averse algorithm is a maximization problem with greater than equal to constraints on the SSD terms, and the mean constraint is implicit in the $h^{th}$ element of the SSD inequality constraints. Also, the BBRS LP is written in terms of Lower Partial Moments. See Bawa (1975) for the relation between the lower partial moment and the second-order stochastic dominance sum of cumulative distributions, $S_{it} = L_{ii} d_{it} = \sum_{x \in \mathcal{X}_{i}} (d_{it} - x_{it}) f_{it}$ for $t = 1, \ldots, h$ and $i = 1, \ldots, N + 1$
Proposition 3: For risk seeking investors, the $\Psi_1$ distribution mean-scale optimal mutually exclusive investment choices and the $\Psi_0$ distribution location-scale optimal portfolio investment choices have equation (11) linear programming solutions greater than one.

In the next section, we show that the seemingly infinite dimensional set of linear programs for the portfolio problem is, on the contrary, finite.

4. Optimal Location-Scale Distribution Choices for All Investors

Bawa (1976, 1977) and Baron (1977) have addressed normally distributed investment choices. We may integrate their results with our findings to identify optimal location-scale distribution investment choices for all investors.

Bawa (1976) was first to identify the normally distributed investment return admissible set for both risk-averse and risk-seeking investors who also prefer more-to-less. His analysis limited investors to portfolio choices. Consistent with the Von Neumann-Morgenstern (VNM) axioms, Baron (1977, pg. 1686) states the following:

“Since one can always construct mixed strategies from pure strategies, by flipping a coin for example, the set of distributions can not be restricted to a set of normal distributions when the objective of the analysis is to order all of the distributions of return that may be generated.”

Baron’s Proposition 4 highlights that risk-seeking individuals will prefer returns generated by mixtures of distributions to those generated from portfolios. Therefore, Bawa’s (1976) admissible portfolio set for risk-seeking individuals is not their optimal set.

Nevertheless, Bawa’s analysis provides a heuristic method for identifying optimal portfolio investments for risk-seeking investors. In applications, his methods may be used to limit the dimensionality of our equation (11) linear programming specification.
In his Theorem 1, Bawa states that risk seeking investor’s will have admissible portfolios “containing at most two securities” from the set of maximum variance securities. Augmenting the portfolio decision space with mixtures, we have, by Baron’s Proposition 4, any two security portfolios dominated by a mixture of the two securities. This mixture is formed by a random draw of one of the two distributions with draw probabilities equal to the portfolio weights.

Furthermore, Baron’s Proposition 3 states that a pure strategy will be preferred to any mixture. An approximation to the optimal set is the individual securities in Bawa’s maximum variance portfolio admissible set. Therefore, the necessary and sufficient optimality criterion of Proposition 3 may be applied only to these candidate choices.10

Bawa’s risk-seeking investor admissible portfolio set is continuous. However, the associated optimal set is discrete. As we have developed in our treatment of mutually exclusive and portfolio investment choices for risk-averse investors, the arguments made for mean-variance optimality extend directly to $\Psi_1$ and $\Psi_0$ location-scale distribution optimality, respectively. Therefore, we state our main result directly:

**Proposition 4:** Investors manifesting non-satiation and risk-preference have an optimal set defined in mean-scale space that is a decreasing discrete function from the maximum mean security choice down to the maximum scale security choice: a subset of the maximum mean-maximum variance admissible set.

a) For risk-averse investors facing mutually exclusive $\Psi_1$ generalized location-scale distribution choices, the optimal set is an increasing discrete step

---

10 Numerically, the Bawa admissible portfolio set elements lie inside an implicit optimal set boundary. As an example, the 0.0038 mean and 0.00476 variance security is dominated by a 0.0038 mean and 0.00606 variance portfolio. In repeated application of our LP (equation 20), the mixture of securities that dominates this portfolio dominates choices with a 0.0038 mean and variance up to 0.00615. As Baron (1977) shows, the mean-maximum variance mixture frontier is concave to the origin and is only sufficient for identifying quadratic utility optimal choices.
function from the minimum mean and scale distribution to the maximum mean distribution that equals the admissible set.

b) For risk-averse investors facing $\Psi_0$ location-scale distribution portfolio choices, the optimal set is the usual mean-scale admissible set. (For normal distributions, the admissible and optimal set is the Markowitz-Tobin-Sharpe mean-variance efficient set. Barry (1974) and Fama (1965) treat T- and Stable distributions, respectively.)

**Corollary:** As noted for admissible sets in Bawa (1976) Theorem 3, if the maximum mean security is also maximum scale, then the optimal set for all investors equals the optimal set for risk-averse investors. (i.e. Risk-seeking investors choose only the maximum mean and scale alternative.)

Taken together our results provide a full characterization of the optimal set for most commonly utilized continuous parametric distribution functions, and for an investor who is either risk-averse or risk-seeking over the entire return domain.

5. Conclusion

For investors manifesting non-satiation and risk-aversion, $U_2$, and who face mutually exclusive generalized location-scale distribution or portfolio location-scale distribution investment returns, we have shown that the second-order stochastic dominance (SSD) admissible set is the Bawa et. al. (1985) optimal set. This optimal set is also the Bawa-Goroff (1982) strictly best set. We conclude that admissible sets of location-scale distributed choice elements are optimal.

In the absence of mean and scale parameter estimation risk, our results highlight Sharpe's (1966) classic mean-variance(scale) risk measure as an optimal delegated financial management choice measure. In this context, a risk-averse portfolio manager
should identify inefficient or dominated choice set elements by this simple mean-scale rule and should not reduce the choice set further before presenting choices to investors.

Peleg-Yaari (1975) and Peleg (1975), Bawa-Goroff (1982), and Dybvig-Ross (1982) show that the SSD admissible set and various “optimal” sets are not, in general, equal. Further analysis of the respective “risk-aversely efficient” and “regular risk-aversely efficient” random variables, “strictly best choices,” and “portfolio efficient sets” is needed. As highlighted by Dybvig-Ross (1982) in the portfolio context and more generally for the mutually exclusive investment choices, non-convex choice sets raise the potential for difficulty in this analysis.

Alternatively, Bawa-Goroff (1982) have shown that the admissible set is dense in the optimal-strictly best set. Therefore, the delegated manager who provides decision makers with admissible choices should not be grossly non-optimal.

In recent work, Kousmanen (2001) and Post (2003) identify portfolio optimal sets through the dual of the distribution-based problem treated here. Instead, the utility functions that select optimal choice elements are identified. Bodurtha (2003) provides an efficient algorithm for their problem and identifies utility functions for optimal mutually exclusive investment choices. To date, this utility-based dual method has not been used to identify optimal choices among general continuous distributions.11

For investors manifesting non-satiation and risk-preference, $U^2$, we modify the Bawa et. al. (1985) risk averse investor algorithm to identify risk-seeking optimal choices. For location-scale distribution choices, the mutually exclusive and portfolio

---

11 For Normal distributions, this analysis is available from the authors. Among the broader location-scale distributions corresponding analysis may not be trivial. For example, the exponential utility function that provides utility-based optimality for normally distributed portfolio choices does not exist for T-predictive distributions – Bawa, Brown and Klein (1976, page 114, footnote 6.)
choice optimal sets are equal. Furthermore, this optimal set contains only individual securities from Bawa’s (1976, 1977) risk-seeking admissible set.

In another financial decision-making context, Dybvig (1990) implemented a first-order stochastic dominance (FSD) admissible set-based analysis to show that some well-known trading strategies are dominated. A natural extension of his analysis is to consider FSD or SSD admissible strategies that may be dominated under CSSD optimality criteria [e.g., Bawa et. al. (1985)].

Motivated by the work of Friedman-Savage (1948) and Kahneman and Tversky (1986), Post-Levy (2003) differentiates between mixed risk-averse and risk-seeking utility functions over discrete distributions. Our results provide cases where dominance of a discrete approximation to standard unbounded choice distributions cannot imply dominance for the true distribution. Location-scale distributions illustrate this caveat's general relevance for determining optimality.

---

12 In the portfolio context, Shalit-Yitzhaki (2000) conduct such an analysis.
Appendix - Lemmas

**Lemma 1:** Scale Dominance Rule

Consider two distributions $F_1$ and $F_2$ in $\Psi_0$ and $\Psi_1$ with finite scales $\sigma_1$ and $\sigma_2$. We set $\sigma_1 < \sigma_2$, then there exist three numerals $x^*, r_1$ and $r_2$, (with $r_1 < r_2$), such that

(I) the density functions $f_1(x)$ and $f_2(x)$ satisfy $f_1(x) < f_2(x)$, if $x < r_1$ or $x > r_2$

(II) the distribution functions have the same value at $x^*$ and satisfy:

\[
F_1(x) < F_2(x) \quad \text{if } x < x^* \\
F_1(x) > F_2(x) \quad \text{if } x > x^*.
\]

**Proof:** The proof has three steps.

Step 1: There are exactly two intersection points for $f_1(x)$ and $f_2(x)$.

The crossing properties of normal, $T$ and Cauchy Generalized Location and Scale distributions, $\Psi_0$ and $\Psi_1$ of Bawa (1975, 1979), are derived from the quadratic functions of the location and scale parameters. In these cases, the densities are, like the normal, functions of a standardized random variable, $(\phi(x)-\mu)/\sigma$. Any two admissible distribution quadratic terms intersect, and the crossing points of the associated densities follow algebraically.\(^{13}\)

Though no analytic generalized stable density functions are available, solution of the Paretian distribution tail approximation to the Stable distribution identifies the

\[^{13}\text{For }T\text{ and Cauchy densities, the roots are } \frac{\mu_i - \mu_j}{\sigma_i - \mu_i} \text{ and } \frac{\mu_i + \mu_j}{\sigma_i + \mu_i}. \text{ For normal densities, the roots are } \frac{\mu_i - \mu_j}{\sigma_i - \mu_i} = \frac{\sqrt{\frac{\sigma_i^2 - \sigma_j^2}{2}}}{\sigma_i^2 - \sigma_j^2} \frac{2}{\sigma_i^2 - \sigma_j^2} \frac{\sigma_i^2 + \sigma_j^2}{\sigma_i^2 - \sigma_j^2} \text{ and } x^* = \frac{\sigma_i^2 - \sigma_j^2}{\sigma_i^2 + \sigma_j^2}.\]
crossing points for two admissible distributions.\(^{14}\) We identify the crossing lower and upper crossing points as \(r_1\) and \(r_2\), respectively.

Step 2: To show (I), let
\[
h(x) \equiv f_1(x) - f_2(x). \tag{A-1}
\]
Following Step 1, it is straightforward to verify that
\[
h'(x) < 0 \quad x \in (-\infty, r_1) \tag{A-2}
\]
\[
h'(x) > 0 \quad x \in (r_2, +\infty) \tag{A-2}
\]
Step 3: To show (II), notice that \(F_1(\infty) = F_2(\infty) = 1\).

Since \(F_1(r_1) = \int_{-\infty}^{r_1} f_1(t) \, dt < \int_{-\infty}^{r_2} f_2(t) \, dt = F_2(r_1)\), and \(\int_{r_2}^{\infty} f_1(t) \, dt < \int_{r_2}^{\infty} f_2(t) \, dt\),

it must be that
\[
\int_{r_1}^{r_2} f_1(t) \, dt > \int_{r_1}^{r_2} f_2(t) \, dt. \tag{A-3}
\]
Both \(F_1(x)\) and \(F_2(x)\) are increasing continuous functions on \((-\infty, \infty)\).\(^{15}\) Therefore, there exists a unique \(x^* \in (r_1, r_2)\), such that
\[
F_1(x^*) = \int_{-\infty}^{x^*} f_1(t) \, dt = \int_{-\infty}^{x^*} f_2(t) \, dt = F_2(x^*), \quad \text{and} \quad F_1(x) < F_2(x) \text{ if } x < x^*. \tag{A-4}
\]

Q.E.D.

\(^{14}\) From Levy (1925) and Nolan (1998), the stable density lower tail is
\[
\phi(x) - \mu \frac{\Gamma(\alpha + \beta) \sin(\pi \alpha)}{\pi \alpha}, \tag{1-1}
\]
and the upper tail is its reflection about the unique mode, see Yamazato (1978). Alternatively, the numerical algorithms of McCulloch (1994) and Nolan (1997) calculate the crossing points.

\(^{15}\) For the generalized distribution cases, \(\Psi_1\), the transforms of \(F_1(x)\) and \(F_2(x)\) are increasing continuous functions on the domain. The density crossing points, \(r_1, r_2\), are defined by location-scale and determined in the transform space. The distribution crossing point is a unique, i.e. the log case \(x^* \in (e^{r_1}, e^{r_2})\), and distribution dominance follows in the return space.
As depicted in Figure 2 panel A, the first part of the Scale Dominance Rule states that the density function curve for the smaller scale distribution, \( F_1 \), always lies below the other one with larger scale, \( F_2 \), on the interval \((-\infty, r_1)\). However, a reversed relationship is true on an interval \((r_2, +\infty)\). The second single distribution crossing point property of the Scale Dominance Rule is depicted in panel B of Figure 2.

**Lemma 2:**

Given two distribution functions, as in Lemma 1, the value of \( F_1(x) \) is negligible compared to the value of \( F_2(x) \) if \( x \) is sufficiently small. More precisely,

\[
\lim_{x \to -\infty} \frac{F_2(x)}{F_1(x)} = +\infty.
\]  

(A-5)

**Proof:** By L'Hopital's Law,

\[
\lim_{x \to -\infty} \frac{F_2(x)}{F_1(x)} = \lim_{x \to -\infty} \frac{F'_2(x)}{F'_1(x)} = +\infty.
\]  

(A-6)

From equation (6),

\[
\lim_{x \to +\infty} \frac{f_2(x)}{f_1(x)} = +\infty.
\]  

(A-7)

Q.E.D.

This Lemma is another interpretation of the Scale Dominance Rule, and states that the distribution curve of larger scale not only dominates the distribution curve with a smaller scale, but also that the magnitude of the latter one is actually negligible. In fact as \( x \to -\infty \), \( F_1(x) \) approaches 0 much faster than \( F_2(x) \) does.
Figure 2
Density- and Distribution-Based Scale Dominance Rules

Panel A: Density-Based

Panel B: Distribution-Based
Lemma 3: The SSD integral (1) is convex.

Proof: The SSD integral is a twice continuously differentiable real-valued function on an open interval. Furthermore, its second derivative is the normal density and hence, non-negative throughout its domain. From Rockafellar (1970), convexity follows by Theorem 4.4, and essentially strict convexity follows by Theorem 26.3 (the SSD integral gradient is the normal distribution and is positive over the real line.)

Lemma 4: A portfolio of $\Psi_0$ location-scale distributed choices SSD dominates the associated mixture of location-scale distributed choices.

Proof: Given Lemma 3 [convexity of the SSD integral (1)], a convex combination (mixture) of these integrals is no less than the SSD integral defined over the linear combination (portfolio) of the associated random variables. Following Baron, the portfolio weight, $\pi$, equals the mixing probability, $\lambda_\pi$.

For a portfolio to CSSD dominate a mixture requires

$$\int_a^x F(\mu_p, \sigma_p, t) dt \leq \lambda_\pi \int_a^x F(\mu_1, \sigma_1, t) dt + (1 - \lambda_\pi) \int_a^x F(\mu_2, \sigma_2, t) dt, \quad \forall x \in (-\infty, \infty) \text{ and } 0 < \alpha < 1.$$  \hfill (A-8)

Defining the portfolio weights to numerically equal the mixture weights, we have

$$x_p = \pi x_1 + (1 - \pi) x_2$$  \hfill (A-9)

$$\mu_p = \pi \mu_1 + (1 - \pi) \mu_2$$

$$\sigma_p^2 = \pi^2 \sigma_1^2 + 2\pi(1 - \pi)\sigma_1\sigma_2\rho + (1 - \pi)^2 \sigma_2^2 \neq \left[ \pi\sigma_1 + (1 - \pi)\sigma_2 \right]^2$$
However, setting the correlation equal to one implies that the portfolio scale is a convex combination of the other two scales, and that this scale deviation is an upper bound on the actual portfolio scale:

\[
\sigma_p \leq \sigma_{p|\rho=1} = \pi \sigma_1 + (1 - \pi) \sigma_2
\]  

Therefore,

\[
\int_a^x \Phi(\mu_p, \sigma_p, t) dt \leq \int_a^x \Phi(\mu_{p|\rho=1}, \sigma_{p|\rho=1}, t) dt
\]  

\[
\leq \alpha \int_a^x \Phi(\mu_1, \sigma_1, t) dt + (1 - \alpha) \int_a^x \Phi(\mu_2, \sigma_2, t) dt,
\]  

\[
\forall x \in (-\infty, \infty) \text{ and } 0 < \alpha < 1.
\]

The first panel of Figure 3 depicts two SSD integrals. The second panel of this figure plots an example for the equally-weighted mixture and portfolio of the two normally distributed choices.
Figure 3
Normal Distribution-Based Second-Order Stochastic Dominance Integrals\textsuperscript{16}

Panel A: Two Alternative Choice Elements

Panel B: Mixture and Portfolio of the Alternatives

\textsuperscript{16} With integration by parts, \[ \int_{-\infty}^{x} F_1(t) \, dt = (x - \mu_i) \Phi \left( \frac{x - \mu_i}{\sigma_i} \right) + \sigma_i \phi \left( \frac{x - \mu_i}{\sigma_i} \right) \, \Phi \left( \frac{x - \mu_i}{\sigma_i} \right) \] and \[ \phi \left( \frac{x - \mu_i}{\sigma_i} \right) \] are the standard normal distribution and density, respectively.
References


Bawa, Vijay S. and Daniel L. Goroff, "Admissible Efficient, and Best Choices under Uncertainty," University of Texas-Austin, Department of Finance working paper 81/82-2-8, 1982.


Yitzhaki, Shlomo and Joram Mayshar, "Characterizing Efficient Portfolios, " Hebrew University of Jerusalem working paper, 1997 (revised 2001.)