Appendix - Normally Distributed Admissible Choices are Optimal

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Abstract

Particular utility functions that are optimizers for a particular admissible set element are identified. The optimality of the normally distributed admissible portfolios is shown in the second-order stochastic dominance integral context.

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For Normal Returns, Mean-Variance Admissible Choices are Optimal

Abstract

Particular utility functions that are optimizers for a particular admissible set element are identified. The optimality of the normally distributed admissible portfolios is shown in the second-order stochastic dominance integral context.

A-1. Optimal Choices among Mutually Exclusive Alternatives

We use a geometric argument to illustrate construction of the utility functions that will choose each element of the SSD admissible set.\(^1\)

Proposition A-1: For every member of the SSD admissible set of choices, there exists a utility function in \(U_2\) that is the optimizer for that utility function.

Proof: Given an admissible set of normal distributions \(\Phi = \{F_1, F_2, \ldots, F_n\}\), we can order the distributions in the admissible set \(\Phi\) in such a way that

\[
\sigma_1 < \sigma_2 < \cdots < \sigma_n, \text{ and } \mu_1 < \mu_2 < \cdots < \mu_n.
\]

Furthermore, these strict relations order the SD admissible set over the real line. Consider any three adjacent distributions, \(F_{i-1}\), \(F_i\), and \(F_{i+1}\), for integer index \(i\), 1 < \(i\) < \(n\). Construct two distributions satisfying the following:

\[
F_{i-\varepsilon} \sim \mathcal{N}\left(\mu_{i-\varepsilon} = \mu_i - \frac{\varepsilon}{2}, \sigma_{i-\varepsilon} = \sigma_i - \varepsilon\right) \text{ and } A-1
\]

\[
F_{i+\varepsilon} \sim \mathcal{N}\left(\mu_{i+\varepsilon} = \mu_i + \varepsilon, \sigma_{i+\varepsilon} = \sigma_i + \frac{\varepsilon}{2}\right)
\]

We can always construct a smooth convex curve (maybe of second-degree) which passes below the mean-variance pairs associated with distributions \(F_{i-\varepsilon}\), \(F_i\), and \(F_{i+\varepsilon}\), and above the mean-variance pairs associated with distributions \(F_{i-1}\) and \(F_{i+1}\). We use this curve as the upper-most equi-utility curve for some utility function. (Other equi-utility curves in the same family should be below this curve.) Since all distributions other than \(F_i\) are below or on the right side of this curve, \(F_i\) is the optimizer of this utility function. Q.E.D.

\(^1\) See Barron (1977) for a thorough treatment of mean-variance utility functions.
For the mutually exclusive investment choice problem, we conclude that the SSD admissible set is the optimal set.

**Alternative Proof of Proposition 1**

**Proposition 1:** Given a set of normal distributions $\Phi = \{F_1, F_2, \cdots, F_n, F_{n+1}\}$, if $\Phi$ is a $\text{U}_2$ admissible set, then it is also the CSSD admissible set and optimal.\(^2\)

**Proof:** $\Phi$ is an admissible set; therefore, distributions are mutually undominated. Since in the normal distribution case, SSD is equivalent to the mean-variance decision rule, we can order the distributions in $\Phi$ in such a way that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_n, \text{ and } \mu_1 < \mu_2 < \cdots < \mu_n.$$ 

The mean and standard deviation of distribution $F_{n+1}$ may be anywhere in the sequence of $F_1, F_2, \cdots, F_n$.

Case 1: $\mu_n = \max_{1 \leq j \leq n} \{\mu_j\} < \mu_{n+1}$. A distribution may only be dominated by an alternative with higher mean. Therefore, the distribution with the highest mean cannot be dominated by a mixture of distribution with lower means.

Case 2: $\mu_{n+1} < \mu_n = \max_{1 \leq j \leq n} \{\mu_j\}$. We divide the set $\Phi$ in two parts: $\Phi_1 = \{F_k, \cdots, F_n\}$, and $\Phi_2 = \{F_{k+1}, \cdots, F_n\}$, such that $\mu_k < \mu_{n+1} < \mu_{k+1}$ and $\sigma_k < \sigma_{n+1} < \sigma_{k+1}$. For CSSD, we require the following:

$$\sum_{j=k+1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t) \, dt + \sum_{j=1}^{k} \lambda_j \int_{-\infty}^{r} F_j(t) \, dt \leq \int_{-\infty}^{r} F_{n+1}(t) \, dt, \forall r, \text{ and } < \text{ some } r \quad (A-2)$$

\(^2\) In the context of previous Generalized Location-Scale distribution family-related footnotes, mean-scale admissible t distributions with the same degree of freedom, stable distributions with the same characteristic exponent and skewness and log-normal distributions are optimal.
Dividing through by the right-hand side of equation (A-2),

\[
\sum_{j=1}^{k} \int_{-\infty}^{r} F_j(t) \, dt + \sum_{j=k+1}^{n} \int_{-\infty}^{r} F_{n+1}(t) \, dt \leq 1, \forall r, \text{ and } r < \text{ some } r
\]

(A-3)

Consider the first term on the left hand side of equation A-3) which is the weighted sum of SSD integrals over distributions 1 to k. By the distribution dominance Lemma 2, all terms in this sum go to zero as r goes to minus infinity.

Analogously, consider the second term on the left hand side of equation A-3) which is the weighted sum of SSD integrals over distributions k+1 to n. By the distribution dominance Lemma 2, all terms in this sum go to infinity as r goes to minus infinity. Furthermore, at least one of the convex combination weights associated with distributions k+1 to n must be greater than zero. Otherwise, the expected value of the dominating mixture will not exceed the expected value of the dominance candidate distribution, which is a necessary dominance condition.

Therefore, the value of the left hand side is far greater than one for sufficiently small r, and approaches infinity as r goes to minus infinity. Admissible normally distributed choice n+1 cannot be dominated under CSSD. \[Q.E.D.\]

A-2. Optimal Portfolio Choices

In an n-security portfolio problem, returns are represented by a random n-vector \( x \), and a choice \( i \) a vector of portfolio investment weights. We assume that the set of all feasible portfolio choices, \( P \), is a closed bounded set:

\[
P = \left\{ \pi = \{\pi_1, \pi_2, \ldots, \pi_n\}^T : \sum_{i=1}^{n} \pi_i = 1, \max_{i} |\pi_i| < M \right\}
\]
Investors have utility functions in the $U_2$ class, and the return vector follows a joint normal distribution. In this case, SSD dominance is equivalent to a mean-variance ordering. As is well known, the efficient set is the upper half of the mean-variance frontier.

Since nothing can dominate the unique portfolio that maximizes the expected value of a $U_2$ function, the SSD admissible set is efficient. Furthermore, we can show that the SSD admissible set (the upper half of the mean-variance frontier) is optimal.

Following Huang and Litzenberger (1988), pp. 64-67:

$$e = E(x), V$$ is the variance-covariance matrix of $x$, $1 = (1, 1, \cdots, 1)^T$.

$$A \equiv I^T V^{-1} e, \quad B \equiv e^T V^{-1} e, \quad C \equiv I^T V^{-1} 1, \quad D \equiv BC - A^2 > 0$$

(A-2)

The equation for the mean-variance efficient frontier is the following:

$$\frac{\sigma^2}{1/C} - \left( \frac{\mu - A}{C} \right)^2 = 1, \quad \frac{A}{C} < \mu < \mu_M, \quad \mu \equiv \pi^T e, \quad \text{and} \quad \sigma^2 \equiv \pi^T V \pi$$

Therefore,

$$\sigma^2 = \frac{1}{C} + \frac{C}{D} \left( \frac{\mu - A}{C} \right)^2$$

(A-3)

**Proposition A-2:** For every member of the SSD admissible portfolio choice set, there exists a utility function in $U_2$ that is the optimizer for that utility function.

**Proof:** To show that a given member of the SSD admissible set is optimal, we construct the associated $U_2$-class utility function. In a one-parameter utility family $\{-e^{-ax}; a > 0\}$ and for a given point on the efficient frontier $\left(\mu, \sigma^2\right)$, we set

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3 See Merton (1972) and Roll (1976) for full treatments of this efficient portfolio set. Our efficient set is the open interval above the minimum variance portfolio and below some arbitrary upper bound. The upper bound is necessary to have a well-posed optimization problem.
\[ a = \frac{D}{C \mu - A} > 0. \]

The associated expected utility is
\[ \mathbb{E}[-e^{-ax}] = \mu - \frac{a \sigma^2}{2} \tag{A-4} \]

Substituting (3) into (4),
\[ \mathbb{E}[-e^{-ax}] = \mu - a \left[ \frac{1}{C} + \frac{C}{D} \left( \mu - \frac{A}{C} \right)^2 \right] \]

This expected utility function will be maximized at
\[ \mu_{\text{max}} = \frac{D}{a C} + \frac{A}{C} = \mu. \quad \text{Q.E.D.} \]

For the portfolio choice problem, we conclude that the mean-variance efficient frontier is both the SSD admissible set and optimal.

### A-3 CSSD Portfolio Choices

For portfolio choices, Baron (1977) has shown that a choice vector, \( \pi \), dominates the associated mixed strategy, \( \lambda \pi \), for all strictly concave von Neumann-Morgenstern utility functions. Yitzhaki-Mayshar (1997) have proven this result in the context of Marginal Stochastic Dominance for general distributions. We present a corollary to these results as Proposition A-3.

We first develop two additional lemmas:

**Lemma A-1**: The SSD integral (1) is convex.

**Proof**: The SSD integral is a twice continuously differentiable real-valued function on an open interval. Furthermore, its second derivative is the normal density and hence, non-negative throughout its domain. From Rockafellar (1970), convexity follows by Theorem 4.4, and essentially strict convexity follows by Theorem 26.3 (the SSD integral gradient is the normal distribution and is positive over the real line.)

**Lemma A-2**: A portfolio of normally distributed choices SSD dominates the associated mixture of normally distributed choices.
Proof: Given Lemma A-1 [convexity of the SSD integral (1)], a convex combination (mixture) of these integrals is no less than the SSD integral defined over the linear combination (portfolio) of the associated random variables. With integration by parts, we have the following:

\[
\int_{-\infty}^{x} F_i(t) \, dt = \sigma_i \left[ \frac{x - \mu_i}{\sigma_i} \right] \Phi \left( \frac{x - \mu_i}{\sigma_i} \right) + \phi \left( \frac{x - \mu_i}{\sigma_i} \right), \text{ and } \Phi \left( \frac{x - \mu_i}{\sigma_i} \right) \text{ and } \phi \left( \frac{x - \mu_i}{\sigma_i} \right) \text{ are the standard normal distribution and density, respectively.}
\]

For a portfolio to CSSD dominate a mixture requires

\[
\sigma_p \left[ \frac{x - \mu_p}{\sigma_p} \right] \Phi \left( \frac{x - \mu_p}{\sigma_p} \right) + \phi \left( \frac{x - \mu_p}{\sigma_p} \right) \leq \alpha \sigma_1 \left[ \frac{x - \mu_1}{\sigma_1} \right] \Phi \left( \frac{x - \mu_1}{\sigma_1} \right) + \phi \left( \frac{x - \mu_1}{\sigma_1} \right) + (1 - \alpha) \sigma_2 \left[ \frac{x - \mu_2}{\sigma_2} \right] \Phi \left( \frac{x - \mu_2}{\sigma_2} \right) + \phi \left( \frac{x - \mu_2}{\sigma_2} \right), \quad \forall x \in (-\infty, \infty) \text{ and } 0 < \alpha < 1.
\]

Defining the portfolio weights to equal the mixture weights, we have

\[
x_p = \alpha x_1 + (1 - \alpha) x_2 \\
\mu_p = \alpha \mu_1 + (1 - \alpha) \mu_2 \\
\sigma_p^2 = \alpha^2 \sigma_1^2 + 2\alpha (1 - \alpha) \sigma_1 \sigma_2 \rho + (1 - \alpha)^2 \sigma_2^2 \neq \left[ \alpha \sigma_1 + (1 - \alpha) \sigma_2 \right]^2
\]

However, setting the correlation equal to one implies that the portfolio standard deviation is a convex combination of the other two standard deviations, and that this standard deviation is an upper bound on the actual portfolio standard deviation:

\[
\sigma_p \leq \sigma_{p|p=1} = \alpha \sigma_1 + (1 - \alpha) \sigma_2
\]

Therefore,

\[
\sigma_p \left[ \frac{x - \mu_p}{\sigma_p} \right] \Phi \left( \frac{x - \mu_p}{\sigma_p} \right) + \phi \left( \frac{x - \mu_p}{\sigma_p} \right) \leq \sigma_{p|p=1} \left[ \frac{x - \mu_p}{\sigma_{p|p=1}} \right] \Phi \left( \frac{x - \mu_p}{\sigma_{p|p=1}} \right) + \phi \left( \frac{x - \mu_p}{\sigma_{p|p=1}} \right) \leq \alpha \sigma_1 \left[ \frac{x - \mu_1}{\sigma_1} \right] \Phi \left( \frac{x - \mu_1}{\sigma_1} \right) + \phi \left( \frac{x - \mu_1}{\sigma_1} \right) + (1 - \alpha) \sigma_2 \left[ \frac{x - \mu_2}{\sigma_2} \right] \Phi \left( \frac{x - \mu_2}{\sigma_2} \right) + \phi \left( \frac{x - \mu_2}{\sigma_2} \right),
\]

\[
\forall x \in (-\infty, \infty) \text{ and } 0 < \alpha < 1.
\]
The first panel of Figure 2 depicts two SSD integrals. The second panel of this figure plots an example for the equally-weighted mixture and portfolio of the two normally distributed choices.

Our CSSD efficient portfolio proposition follows:

**Proposition A-3:** The mean-variance efficient frontier choices are CSSD admissible.

**Proof:** Given Lemma A-2, any mixture of alternatives is dominated by an associated portfolio. Any portfolio not associated with the mean-variance efficient frontier is dominated by some element of the set of portfolios on the efficient frontier. Therefore, mean-variance efficient portfolio choices dominate mixtures of portfolio distributions, and all such portfolios are CSSD admissible.

Like mutually exclusive choice CSSD Proposition 1, Proposition A-3 shows that the entire mean-variance efficient portfolio frontier is optimal.\(^4\)

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\(^4\) Analogous results will hold for \(t\) distributions with the same degree of freedom, Cauchy distributions, and stable distributions with the same characteristic exponent and skewness parameter. Since portfolios of log-normal returns are not log-normal, the mutually exclusive choice result for this distribution does not extend similarly.
With integration by parts, \( \int_{-\infty}^{x} F_1(t) \, dt = (x - \mu_i) \Phi \left( \frac{x - \mu_i}{\sigma_i} \right) + \sigma_i \phi \left( \frac{x - \mu_i}{\sigma_i} \right) \Phi \left( \frac{x - \mu_i}{\sigma_i} \right) \) and

\( \phi \left( \frac{x - \mu_i}{\sigma_i} \right) \) are the standard normal distribution and density, respectively.
Additional References


