Two-Country Segmented and Partially Segmented Market Cross-section Regression (CSR) Test Specification and Inference

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Abstract

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Two-Country Segmented and Partially Segmented Market Cross-section Regression (CSR) Test Specification and Inference

Abstract: Empirical methods of Shanken (1992) are extended to the case in which different factors are related to particular countries' security expected returns. We show that Shanken's "three-pass Gauss-Newton" estimator is both consistent and asymptotically efficient. This estimator is analogous to Gibbon's (1982) multivariate regression method (MVRM), and heteroskedasticity and autocorrelation adjustments to this method are discussed. As an alternative to full maximum likelihood estimation or its robust variants, the linearized method that we develop will, minimally, be useful for both determining reasonable parameter starting values and for pre-screening higher dimensional model alternatives. We also fimd that Shanken's consistent and asymptotically efficient integrated market two-pass covariance estimator is not asymptotically efficient in the segmented markets case.

From the inference perspective, the segmented market case is most complicated and computationally intensive. Therefore, we treat this case in detail. Any homogeneity imposed in the partially segmented case results in simplified estimators. Importantly, only in the fully integrated market case are the Shanken (1992) two-pass estimators efficient.

We develop the model in five sections. Section one defines our (rather dense) notation. Section two specifies the two country segmented market maximum likelihood estimator. In section three, we specify cross-section regression estimators through a theorem and a set of corollaries. In section four, a standard heteroskedasticity and autocorrelation adjusted estimator is defined. Section five concludes this work.

1. Definitions and Preliminaries

In period t, the country g - asset h return error is the following:

$$e_{ght} = R_{ght} - g_{0_g}\left(1 - \sum_{j=1}^{k_{2_g}} b_{2_g hj}\right) - \sum_{j=1}^{k_{2_g}} b_{2_g hj} F_{2_g tj} - \sum_{j=1}^{k_{1_g}} b_{1_g hj}\left[F_{1_g tj} - m_{1_g j} + g_{1_g j}\right]$$

$$\forall g \in [1, G=2],\ h \in [1,n] \text{ and } t \in [1,T] \qquad 1)$$

k_{1_g} and k_{2_g} are the number of non-traded and traded factors for country group g, respectively.

The following assumptions are made: the errors are correlated cross-sectionally, but uncorrelated across time periods, the errors are uncorrelated with the factor innovations, the factor innovations are uncorrelated across time, and both errors and factor innovations are distributed multivariate normal.

Define $g^*_{1_g j} = g_{1_g j} - m_{1_g j}$, across securities, <u>two</u> countries (g = 1 or 2) and time, we have

$$e = R - g_0 \otimes i_{nT} - X_b\, b\ , \qquad 2nT \ x\, 1\ , \quad e \sim N(0, \Sigma \otimes I_T\,),\ S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \qquad 2)$$

$$X_b = \begin{bmatrix} I_n \otimes \left[\left(F_{1_1} + i_T \otimes g_{1_1}^{*\prime} \right) : \left(F_{2_1} - i_T \otimes g_{0_1} i'_{k_{2_1}} \right) \right] & 0_{nT \; x \; n\left(k_{1_2} + k_{2_2}\right)} \\ 0_{nT \; x \; n\left(k_{1_1} + k_{2_1}\right)} & \left[I_n \otimes \left(F_{1_2} + i_T \otimes g_{1_2}^{*\prime} \right) : \left(F_{2_2} - i_T \otimes g_{0_2} i'_{k_{2_2}} \right) \right] \end{bmatrix},$$

a $2nT \; x \; n \sum_{i=1}^{2} \left(k_{1_i} + k_{2_i} \right)$ matrix.

$R_{gh} = \left[R_{gh1}, \ldots R_{gnT} \right]', \quad T \; x \; 1$

$R_g = \left[R'_{g1}, \ldots, R'_{gn} \right], \qquad nT \; x \; 1$

$R = \left[R'_1 : R'_2 \right]', \qquad 2nT \; x \; 1, \; R \sim N\left(E \otimes I_t, [S + bDb'] \otimes I_t \right)$

$g_0 = [g_{01}, g_{02}]', \qquad 2 \; x \; 1$

$F_{i_g t} = \left[F_{i_g t1}, \ldots, F_{i_g t k_{1_g}} \right], \qquad 1 \; x \; k_{i_g} \quad i = 1 \, or \, 2$

$F_{i_g} = \left[F'_{i_g 1}, \ldots, F'_{i_g T} \right]', \quad T \; x \; k_{i_g}, \quad F'_{i_g t} \sim N(m_{i_g}, D_{i_g i_g}), \quad i = 1 \text{ or } 2, \; g = 1 \text{ or } 2$

$F_g = \left[(F_{1_g} - m'_{1_g} \otimes i_T) : (F_{2_g} - m'_{2_g} \otimes i_T) \right], \; T \; x \; (k_{1_g} + k_{2_g}), \; F_g \sim N\left(0, D_{gg} \right), D_{gg} = \begin{bmatrix} D_{1_g 1_g} & D_{1_g 2_g} \\ D_{2_g 1_g} & D_{2_g 2_g} \end{bmatrix}$

$F = \left[F_1 : F_2 \right], \; T \; x \; \sum_{i=1}^{2} \left(k_{1_i} + k_{2_i} \right), \; F \sim N\left(0, D \right), D = \begin{bmatrix} D_{11} & D_{12} \\ D_{12}' & D_{22} \end{bmatrix}, \; D_{12} = \begin{bmatrix} D_{1_1 1_2} & D_{1_1 2_2} \\ D_{2_1 1_2} & D_{2_1 2_2} \end{bmatrix}$

$g_{1_g} = \left[g_{1_g 1}, \ldots, g_{1_g k_{1_g}} \right]', \qquad k_{1_g} x1$

$g_{1_g}^{*} = g_{1_g} - m_{1_g}, \qquad k_{1_g} x1$

$G_1^{*} = \left[g_{0_1}, g_{1_1}^{*\prime}, g_{0_2}, g_{1_2}^{*\prime} \right]', \quad \left(2 + k_{1_1} + k_{1_2} \right) x1$

$g_{2_g} = m_{2_g} - g_{0_g} i_{k_{2_g}}, \qquad k_{2_g} x1$

$G_g = \left[g'_{1_g}, g'_{2_g} \right]', \quad \left(k_{1_g} + k_{2_g} \right) \; x \; 1$

$G = \left[g_{0_1}, G'_1, g_{0_2}, G'_2 \right]', \quad \left(2 + \sum_{i=1}^{2} \left(k_{1_i} + k_{2_i} \right) \right) \; x \; 1$

$b_{i_g h} = \left[b_{i_g h1}, \ldots, b_{i_g h k_{i_g}} \right], \qquad 1 x k_{i_g}$

$b_{i_g} = \left[b'_{i_g 1}, \ldots, b'_{i_g n} \right]', \qquad n x k_{i_g}$

$$b_g = \left[b_{1_g 1}, b_{2_g 1}, b_{1_g 2}, b_{2_g 2}, ..., b_{1_g n}, b_{2_g n} \right]', \qquad n\left(k_{1_g} + k_{2_g}\right) x 1$$

$$b = \left[b_1' \; b_2' \right]', \qquad n \sum_{i=1}^{2} \left(k_{1_i} + k_{2_i}\right) x \, 1$$

As we estimate both "β" and "Γ" terms, an alternative definition of the error vector is useful:

$$e = R - \begin{bmatrix} I_n \otimes \left[F_{1_1} : F_{2_1} \right] & 0_{nT \, x \, n\left(k_{2_1} + k_{2_2}\right)} \\ 0_{nT \, x \, n\left(k_{1_1} + k_{2_1}\right)} & I_n \otimes \left[F_{2_1} : F_{2_2} \right] \end{bmatrix} \; b - X_G G_1^* \qquad 3)$$

$$X_G = \begin{bmatrix} \left[\left(i_n - b_{2_1} i_{k_{2_1}} \right) : b_{1_1} \right] \otimes i_T & 0_{nT \, x \, n\left(k_{1_2} + 1\right)} \\ 0_{nT \, x \, n\left(k_{1_1} + 1\right)} & \left[\left(i_n - b_{2_2} i_{k_{2_2}} \right) : b_{1_2} \right] \otimes i_T \end{bmatrix}$$

2. Maximum Likelihood Estimates

Under a joint normality assumption, the log likelihood function is

$$L\left(b, \; G_1^*, S\right) \propto -\frac{T}{2}|S| - \frac{1}{2}\left[e'\left(S^{-1} \otimes I_T\right) e \right] \qquad 4)$$

From the first order conditions,

$$b^{mle} = \left(X_b{}' \left(S^{-1} \otimes I_T\right) X_b \right)^{-1} X_b{}' \left(S^{-1} \otimes I_T\right)\left(R - g_0 \otimes i_{nT}\right) \qquad 5)$$

$$G_1^{*mle} = \left(X_G' \left(S^{-1} \otimes I_T\right) X_G \right)^{-1} X_G{}' \left(S^{-1} \otimes I_T\right) \left(R - \begin{bmatrix} I_n \otimes \left[F_{1_1} : F_{2_1} \right] & 0_{nT \, x \, n\left(k_{2_1} + k_{2_2}\right)} \\ 0_{nT \, x \, n\left(k_{1_1} + k_{2_1}\right)} & I_n \otimes \left[F_{2_1} : F_{2_2} \right] \end{bmatrix} b \right)$$

The second order conditions are the following:

$$¶ L / ¶b¶b' = \left(X_b' \left(S^{-1} \otimes I_T\right) X_b \right) \qquad 7)$$

$$¶ L \big/ ¶b¶G_1^* = \left(X_G' \left(S^{-1} \otimes I_T\right) X_b \right)$$

$$¶ L \big/ ¶G_1^*¶G_1^{*\prime} = \left(X_G' \left(S^{-1} \otimes I_T\right) X_G \right)$$

For the parameter vector of interest, $q = \left[b', G_1^{*'} \right]'$, the covariance matrix follows as the inverse of the information matrix:

$$V(q) = \begin{bmatrix} \hat{X}_b' \left(\hat{S}^{-1} \otimes I_T \right) \hat{X}_b & \hat{X}_b' \left(\hat{S}^{-1} \otimes I_T \right) \hat{X}_G \\ \hat{X}_G' \left(\hat{S}^{-1} \otimes I_T \right) \hat{X}_b & \hat{X}_G' \left(\hat{S}^{-1} \otimes I_T \right) \hat{X}_G \end{bmatrix}^{-1} \qquad 7)$$

3. Cross-Section Regression-based Estimation

First-pass beta estimates, $\hat{b}^{1st}$, follow from

$$e^{1st} = R - \bar{R} \otimes i_T - \begin{bmatrix} I_n \otimes \left[\left(F_{1_1} : F_{2_1} \right) - \left(\bar{F}_{1_1} : \bar{F}_{2_1} \right) \otimes i_T \right] & 0_{nT\,x\left(k_{2_1}+k_{2_2}\right)} \\ 0_{nT\,x\left(k_{1_1}+k_{1_2}\right)} & I_n \otimes \left[\left(F_{1_2} : F_{2_2} \right) - \left(\bar{F}_{1_2} : \bar{F}_{2_2} \right) \otimes i_T \right] \end{bmatrix} b^{1st} \qquad 8)$$

$\bar{R} = \left(i_T' \otimes I_{2n} \right) R$, and $\bar{F}_{i_g} = F_{i_g}' i_T / T$, $1\ x\ k_{i_g}$, and the usual properties hold.

Second-pass gamma estimates will follow by averaging over 3), multiplying by $I_{2n} \otimes \iota_T' T^{-1}$, and substituting in the first-pass beta estimates:

$$\bar{R} = \begin{bmatrix} I_n \otimes \bar{F}_{2_1} & O_{n\,x\,nk_{2_2}} \\ O_{n\,x\,nk_{2_1}} & I_n \otimes \bar{F}_{2_2} \end{bmatrix} \begin{bmatrix} \hat{b}_{2_1 1}^{1st'} \\ \vdots \\ \hat{b}_{2_2 n}^{1st'} \end{bmatrix} + \hat{\bar{X}}_G \bar{G}_1 + \bar{y} \qquad 9)$$

$$\bar{y} = \bar{e} - \begin{bmatrix} I_n \otimes \left[\bar{g}_{1_1}' : \bar{g}_{2_1}' \right] & O_{n\,x\,n\left(k_{1_2}+k_{2_2}\right)} \\ O_{n\,x\,n\left(k_{1_1}+k_{2_1}\right)} & I_n \otimes \left[\bar{g}_{1_2}' : \bar{g}_{2_2}' \right] \end{bmatrix} u \qquad 10)$$

$$u = vech\left(\hat{b}^{1st} - b \right) = X\ e^{1st}$$

$$X = \begin{bmatrix} I_n \otimes \left(\hat{F}_1' \hat{F}_1 \right)^{-1} \hat{F}_1' & O_{n\left(k_{1_1}+k_{1_2}\right)x\,nT} \\ O_{n\left(k_{2_1}+k_{2_2}\right)x\,nT} & I_n \otimes \left(\hat{F}_2' \hat{F}_2 \right)^{-1} \hat{F}_2' \end{bmatrix}$$

$$\hat{F}_g = \left[\left(F_{1_g} - \bar{F}_{1_g} \otimes i_T \right) : \left(F_{2_g} - \bar{F}_{2_g} \otimes i_T \right) \right]$$

$$\bar{g}_{1_g} = g_{1_g} + \bar{F}_{1_g}' - m_{1_g}' = g_{1_g}^{*} + \bar{F}_{1_g}', \text{and } \bar{g}_{2_g} = \bar{F}_{2g}' - g_{0_g} i_{k_{2_g}}$$

Lemma 1 - Under our joint normality assumption, and conditional on the factor vector innovations, F, all of U, $\bar{e}$, and $\hat{b}$ are independent.

Proof -
$$E(U\bar{\varepsilon}|F) = E\left[\Xi\varepsilon\varepsilon'[I_{2n}\otimes\iota_T]\frac{1}{T}\right] = \Xi\left[\Sigma\otimes I_T\right][I_{2n}\otimes\iota_T]\frac{1}{T}$$
$$= \Xi\left[\Sigma\otimes 1_T\right]/T = 0$$

As $\hat{F}_i'\iota_T\sigma_{ijk} = 0,$ for all i = 1, 2, and j, k = 1,..., n.

Since $E(\varepsilon_{it}\delta_{ks}) = 0$, for all i, k, t, s, then $E(U\bar{\varepsilon}') = 0$. Since

$COV(U\bar{\varepsilon}|F) = COV(U,\bar{\varepsilon}) = 0$, implying independence under normality, and as

$E(\bar{\varepsilon}|F) = E(U|F) = 0$, then $COV(\bar{\varepsilon}, F) = COV(U, F) = 0$. Q.E.D.

Lemma 2 - $E\left(\hat{\varepsilon}^{1st'}\bar{\varepsilon}^{1st}\right) = 0_{2nT,2n}$

Proof - $\hat{e}_{gh}^{1st} = R_{gh} - F_g\left(F_gF_g\right)^{-1}F_g\,R_{gh} = M_gR_{gh}$ (conserving notation the RHS unit vector is omitted). $M_g = I_T - F_g\left(F_g'F_g\right)^{-1}F_g',$ which is a T x T idempotent matrix. Note also that both $M_gF_g = 0_T$ and $M_gi_T = 0_T$.

As $R_{gh} = F_g b_{gh} + e_{gh}$, $\hat{e}_{gh} = M_g e_{gh}$. Across assets, we have

$$E\left[\hat{e}^{1st'}\bar{e}^{1st}\right] = E\left[\begin{bmatrix} I_n\otimes M_1 & I_n\otimes O_{T,T} \\ I_n\otimes O_{T,T} & I_n\otimes M_2\end{bmatrix} e^{1st}e^{1st'}\left(I_{2n}\otimes i_T\right)T^{-1}\right]$$
$$= \begin{bmatrix} I_n\otimes M_1 & I_n\otimes O_{T,T} \\ I_n\otimes O_{T,T} & I_n\otimes M_2\end{bmatrix}\left(\Sigma\otimes I_T\right)\begin{bmatrix} I_n\otimes\iota_T & I_n\otimes O_T \\ I_n\otimes O_T & I_n\otimes\iota_T\end{bmatrix}T^{-1}$$
$$= \begin{bmatrix} S_{11}\otimes M_1 i_T & S_{12}\otimes M_1 i_T \\ S_{21}\otimes M_2 i_T & S_{22}\otimes M_2 i_T\end{bmatrix}T^{-1} = S\otimes 0_T T^{-1} = 0_{2nT,2n}$$ Q.E.D.

Lemma 3 - $E\left[\bar{y}e'|F\right] = \left(S\otimes i_T'\right)T^{-1}$

Proof - $E\left[\bar{y}\varepsilon'|F\right] =$

$$E\left[\left[T^{-1}\left(I_{2n}\otimes i_T'\right) - \begin{bmatrix} I_n\otimes\left[\bar{g}_{1_1}' : \bar{g}_{2_1}'\right]\left(\hat{F}_1'\hat{F}_1\right)^{-1}\hat{F}_1' & O_{n\,x\,nT} \\ O_{n\,x\,nT} & I_n\otimes\left[\bar{g}_{1_2}' : \bar{g}_{2_2}'\right]\left(\hat{F}_2'\hat{F}_2\right)^{-1}\hat{F}_2'\end{bmatrix}\right]ee'\right]$$

$= \left(S\otimes i_T'\right)T^{-1}$, following Lemma 1. Q.E.D.

3.1 Three-pass "Gauss-Newton" Estimation

Following Gibbons (1982) and Shanken (1985, 1992), we also define a three pass-linearized Gauss-Newton estimator, which is based on first pass OLS beta (β) estimates and second-pass gamma (Γ*) estimates.

A Gauss-Newton approximation is $\Gamma_{GN}\beta \cong \Gamma_{GN}\hat{\beta} + \hat{\Gamma}_{GN}\beta - \hat{\Gamma}_{GN}\hat{\beta}$

$$G_{GN} = \begin{bmatrix} I_n \otimes i_T g_{1_1}^{*\prime} & 0_{nT\,x\,n\left(k_{2_1}+k_{2_2}\right)} \\ 0_{nT\,x\,n\left(k_{1_1}+k_{2_1}\right)} & I_n \otimes i_T g_{1_2}^{*\prime} \end{bmatrix} \tag{11}$$

On substitution and rearrangement, $R_{GN} = R - \hat{g}_o \otimes i_{nT} + \hat{X}_b\,\hat{b} \cong \hat{X}_b\,b + \hat{X}_\Gamma \Gamma_1^* + e_{GN} = \hat{Z}q + e_{GN}$

Define $Z = \left[\hat{X}_b : \hat{X}_G\right]$, a $2nT\ x\ \left[n\sum_{i=1}^{2}\left(k_{1_i}+k_{2_i}\right)+2+\sum_{i=1}^{2}k_{i_i}\right]$ matrix.

The GLS (three-pass) estimator of $q = \left[b', G'\right]'$ is

$$\hat{q} = \left(Z'\left(S^{-1}\otimes I_T\right)Z\right)^{-1} Z'\left(S^{-1}\otimes I_T\right)R_{GN} \tag{12}$$

We state our primary inference result as an extension of Shanken's Gauss-Newton Appendix (1992):

Theorem 1 - The three-pass Gauss-Newton estimator is consistent and asymptotically efficient.

Proof: Lemmas 1, 2 and 3 imply asymptotic independence of the errors in beta estimates, the errors in gamma estimates and the innovations (ε). As $\hat{b}$, $\hat{X}_G$ and $\hat{S}$ are, therefore, all consistent, $\hat{q}$ is consistent.

$$V\left(\hat{q}\right) = \left(Z'\left(S^{-1}\otimes I_T\right)Z\right)^{-1} = \begin{bmatrix} V(b,b) & V(b,G_1^*) \\ V(G_1^*,b) & V(G_1^*,G_1^*) \end{bmatrix}, \tag{13}$$

and $$\left(Z'\left(S^{-1}\otimes I_T\right)Z\right) = \begin{bmatrix} \hat{X}_b'\left(\hat{S}^{-1}\otimes I_T\right)\hat{X}_b & \hat{X}_b'\left(\hat{S}^{-1}\otimes I_T\right)\hat{X}_G \\ \hat{X}_G'\left(\hat{S}^{-1}\otimes I_T\right)\hat{X}_b & \hat{X}_G'\left(\hat{S}^{-1}\otimes I_T\right)\hat{X}_G \end{bmatrix}$$

This matrix converges to the maximum likelihood information matrix, and the estimator is asymptotically efficient. Q.E.D.

The lower right block of the parameter covariance matrix is defined as follows:

$$V(G_1^*,G_1^*)=\begin{bmatrix} v(g_{0_1},g_{0_1}) & v(g_{0_1},g_{1_1}^{*\prime}) & v(g_{0_1},g_{0_2}) & v(g_{0_1},g_{1_2}^{*\prime}) \\ v(g_{1_1}^{*},g_{0_1}) & V(g_{1_1}^{*},g_{1_1}^{*\prime}) & v(g_{1_1}^{*},g_{0_2}) & V(g_{1_1}^{*},g_{1_2}^{*\prime}) \\ v(g_{0_2},g_{0_1}) & v(g_{0_2},g_{1_1}^{*\prime}) & v(g_{0_2},g_{0_2}) & v(g_{0_2},g_{1_2}^{*\prime}) \\ v(g_{1_2}^{*},g_{0_1}) & V(g_{1_2}^{*},g_{1_1}^{*\prime}) & v(g_{1_2}^{*},g_{0_2}) & V(g_{1_2}^{*},g_{1_2}^{*\prime}) \end{bmatrix}$$

Based on these definitions, we state the covariance matrix for all prices of risk, Γ:

$$V(G,G)=\begin{bmatrix} v(g_{0_1},g_{0_1}) & v(g_{0_1},g_{1_1}^{*\prime}) & -v(g_{0_1},g_{0_1})\otimes i'_{k_{2_1}} & v(g_{0_1},g_{0_2}) & v(g_{0_1},g_{1_2}^{*}) & -v(g_{0_1},g_{0_2})\otimes i'_{k_{2_2}} \\ v(g_{1_1}^{*},g_{0_1}) & V(g_{1_1}^{*},g_{1_1}^{*\prime}) & -v(g_{1_1}^{*},g_{0_1})\otimes i'_{k_{2_1}} & v(g_{1_1}^{*},g_{0_2}) & V(g_{1_1}^{*},g_{1_2}^{*}) & -v(g_{1_1}^{*},g_{0_2})\otimes i'_{k_{2_2}} \\ -v(g_{0_1},g_{0_1})\otimes i_{k_{2_1}} & -v(g_{0_1},g_{1_1}^{*\prime})\otimes i_{k_{2_1}} & v(g_{0_1},g_{0_1})\otimes i_{k_{2_1}} i'_{k_{2_1}} & -v(g_{0_1},g_{0_2})\otimes i_{k_{2_1}} & -v(g_{0_1},g_{1_2}^{*})\otimes i_{k_{2_1}} & v(g_{0_1},g_{0_2})\otimes i_{k_{1_1}} i'_{k_{2_2}} \\ v(g_{0_2},g_{0_1}) & v(g_{0_2},g_{1_1}^{*\prime}) & -v(g_{0_1},g_{0_1})\otimes i'_{k_{2_1}} & v(g_{0_2},g_{0_2}) & v(g_{0_2},g_{1_2}^{*}) & -v(g_{0_2},g_{0_2})\otimes i'_{k_{2_2}} \\ v(g_{1_2}^{*},g_{0_1}) & V(g_{1_2}^{*},g_{1_1}^{*\prime}) & -v(g_{1_2}^{*},g_{0_1})\otimes i'_{k_{2_1}} & v(g_{1_2}^{*},g_{0_2}) & V(g_{1_2}^{*},g_{1_2}^{*}) & -v(g_{0_2},g_{1_2}^{*})\otimes i'_{k_{2_2}} \\ -v(g_{0_1},g_{0_2})\otimes i_{k_{2_2}} & -v(g_{0_2},g_{1_1}^{*\prime})\otimes i_{k_{2_2}} & v(g_{0_1},g_{0_2})\otimes i_{k_{2_2}} i'_{k_{2_1}} & -v(g_{0_2},g_{0_2})\otimes i_{k_{2_2}} & -v(g_{0_2},g_{1_2}^{*})\otimes i_{k_{2_2}} & v(g_{0_2},g_{0_2})\otimes i_{k_{2_2}} i'_{k_{2_2}} \end{bmatrix}$$

$$+\,T^{-1}\begin{bmatrix} 0 & 0_{1x(k_{1_1}+k_{2_1})} & 0 & 0_{1x(k_{1_2}+k_{2_2})} \\ 0_{(k_{1_1}+k_{2_1})x1} & D_{11} & 0_{(k_{1_1}+k_{2_1})x1} & D_{12} \\ 0 & 0_{1x(k_{1_1}+k_{2_1})} & 0 & 0_{1x(k_{1_2}+k_{2_2})} \\ 0_{(k_{1_2}+k_{2_2})x1} & D_{21} & 0_{(k_{1_2}+k_{2_2})x1} & D_{22} \end{bmatrix} \qquad 14)$$

3.2 Two-pass Estimation for the Two Country Case

Based on two-pass sample gamma estimates, we define the second-pass regression equation error:

$$e \;=\; \bar{R}^{2nd} - \hat{\bar{X}}_G\,\hat{\bar{G}}_1 \qquad 15)$$

$$\bar{R}^{2nd} \;=\; \bar{R} - \begin{bmatrix} I_n \otimes \bar{F}_{2_1} & O_{n\,x\,n\,k_{2_2}} \\ O_{n\,x\,n\,k_{2_1}} & I_n \otimes \bar{F}_{2_2} \end{bmatrix} \begin{bmatrix} \hat{b}^{1st\prime}_{2_1 i} \\ \vdots \\ \hat{b}^{1st\prime}_{2_2 n} \end{bmatrix}$$

This error forms the basis of the Shanken (1985) "Linearity Test." This statistic is a quadratic form and is of the test class of the Hotelling T^2 type:

$$Q^L \propto T \quad e' S^{-1} e \tag{16}$$

The proportional relation is stated to reflect Shanken's subsequent scaling of the statistic to induce centrality.

To analyze this statistic in the partially segmented and segmented market case, we follow Shanken directly. Define $\hat{A} = \left(\hat{\bar{X}}_G' \hat{S}^{-1} \hat{\bar{X}}_G' \right)^{-1} \hat{\bar{X}}_G' \; S^{-1}$, so that $\hat{\bar{G}} = \hat{A}\bar{R}^{2nd}$. Set $\hat{B} = I_{2n} - \hat{\bar{X}}_G \hat{A}$. Also note $\hat{A}\hat{\bar{X}}_G = I_{2n}$, $\hat{B}\hat{\bar{X}}_G = 0_{2n\,x\,2n}$ and $\hat{B}$ is idempotent of rank B_r, $B_r = 2n - \sum_{g=1}^{2} \left(k_{1_g} + k_{2_g} \right)$. Then,

$$e = \hat{B}\bar{R}^{2nd} = \hat{B}\bar{y} \tag{17}$$

To determine potential adjustments, define (asymptotically so that $\hat{B}'\hat{\Sigma}^{-1}\hat{B}$ is replaced by $B'\Sigma^{-1}B$):

$$Q = Te'S^{-1}e = T\bar{y}'B'S^{-1}By = e'Qe$$

$$Q = \begin{bmatrix} I_n \otimes F_1 & I_n \otimes O_T \\ I_n \otimes O_T & I_n \otimes F_2 \end{bmatrix} B'S^{-1}B \begin{bmatrix} I_n \otimes F_1' & I_n \otimes O_T' \\ I_n \otimes O_T' & I_n \otimes F_2' \end{bmatrix} \tag{18}$$

where $F_i = \left[i_T - TF_i \left(F_i' F_i \right)^{-1} \hat{\bar{G}}_1 \right] / \sqrt{T}$.

Again, analogous to Shanken (1985):

Corollary 1 - Given the results of Lemmas 1, 2 and 3 (and conditional on F),

$$Q = T\bar{y}'B'\Sigma^{-1}B\bar{y} \sim \chi^2\left(B_r, \mu_Q\right)$$

$$m_Q = Tu' \begin{bmatrix} I_n \otimes \hat{\bar{G}}_1 & O_{n\,x\,n\left(k_{2_1}+k_{2_2}\right)} \\ O_{n\,x\,n\left(k_{1_1}+k_{1_2}\right)} & I_n \otimes \hat{\bar{G}}_2 \end{bmatrix} B'S^{-1}B \begin{bmatrix} I_n \otimes \hat{\bar{G}}_1 & O_{n\,x\,n\left(k_{2_1}+k_{2_2}\right)} \\ O_{n\,x\,n\left(k_{1_1}+k_{1_2}\right)} & I_n \otimes \hat{\bar{G}}_2 \end{bmatrix} u$$

Proof - To derive the distribution of Q when $\bar{g}_1 \neq \bar{g}_2$, we can follow Shanken (1985) directly. (See Graybill (1961), Theorem 4.9, p. 84, also). Asymptotically as $\hat{b}_i = b_i$ and $\hat{S} = S$, and from equation 16),

$$\tilde{y} \sim N\left(-\sqrt{T}\begin{bmatrix} I_n \otimes \hat{\bar{G}}_1 & O_{n\,x\,n\left(k_{2_1}+k_{2_2}\right)} \\ O_{n\,x\,n\left(k_{1_1}+k_{1_2}\right)} & I_n \otimes \hat{\bar{G}}_2 \end{bmatrix} u, S \right)$$

And Q is noncentral-χ^2 conditional on F. Q.E.D.

The location parameter itself is a random variable. Therefore, the distribution of this statistic under the null should be generated by Monte Carlo methods.

In the integrated markets case of one asset group (G = 1), Shanken has shown that this statistic can be "centered".

We restate Shanken's result in our notation:

Corollary 2 - When a common set of factor prices is the maintained hypothesis, $(F_{1_1} = F_{1_2}$, and $F_{2_1} = F_{2_2})$ and $(\hat{\bar{g}}_{1_1} = \hat{\bar{g}}_{1_2}$, and $\hat{\bar{g}}_{2_1} = \hat{\bar{g}}_{2_2})$ as in Shanken (1985) and Shanken and Weinstein (1990), the test statistic Q is distributed $\Phi_1' \Phi_1 \chi^2(B_r)$.

Proof - The matrix in equation 14) reduces to the following:

$$[I_{2n} \otimes F_1]\left[B' S^{-1} B\right][S \otimes F_1' F_1] B' S^{-1} B [S \otimes I_T][I_{2n} \otimes F_1'] \qquad 20)$$

$$= [I_{2n} \otimes F_1] B' S^{-1} B [S \otimes I_T][I_{2n} \otimes F_1'][F_1' F_1]$$

As $F_1' F_1$ is a scalar, and $[S \otimes I_T][I_{2n} \otimes F_1'] = [I_{2n} \otimes F_1'][S \otimes I_T]$, the following matrix is idempotent:

$$\frac{Q\,(S \otimes I_T)}{F_1' F_1}$$

Therefore, from 14), and Theorem 4.8 in Graybill (1961) (with B_r being the rank of the quadratic form-weighting matrix),

$$Q \sim F_1' F_1 c^2(B_r), \text{ or } \frac{Q}{F_1' F_1} \sim c^2(B_r) \qquad 21)$$

As $\bar{g}_{i_g} = g_{i_g} \ \forall i, g$ asymptotically, $F_1'F_1 = 1 + [g_{1_1}' : g_{2_1}'] \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}^{-1} \begin{bmatrix} g_{1_1} \\ g_{2_1} \end{bmatrix}$. Q.E.D.

We show that the "segmented" market statistic of Corollary 1 cannot be centered.

Theorem 2 - If factor prices are not equal across asset classes, then the Q statistic 18) is not distributed central-χ^2.

Proof - From Graybill (1961), Theorem 4.8 pg. 83, Q will be distributed central-χ^2 if and only if the matrix $\Theta(\Sigma \otimes I_T)$ is idempotent. We posit this matrix to be idempotent:

$$Q(S \otimes I_T)Q(S \otimes I_T) = Q(S \otimes I_T) \qquad 22)$$

Break out $\Sigma \otimes I_T$ as $\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} \otimes I_T$, and

$$\left[Q \begin{bmatrix} S_{11} \otimes I_T & S_{12} \otimes I_T \\ S_{12} \otimes I_T & S_{22} \otimes I_T \end{bmatrix}\right] \left[Q \begin{bmatrix} S_{11} \otimes I_T & S_{12} \otimes I_T \\ S_{12} \otimes I_T & S_{22} \otimes I_T \end{bmatrix}\right] =$$

$$\begin{bmatrix} I_n \otimes F_1 & I_n \otimes O_T \\ I_n \otimes O_T & I_n \otimes F_2 \end{bmatrix} B'S^{-1}B \begin{bmatrix} S_{11} \otimes F_1'F_1 & S_{12} \otimes F_1'F_2 \\ S_{22} \otimes F_2'F_1 & S_{22} \otimes F_2'F_2 \end{bmatrix} B'S^{-1}B \begin{bmatrix} S_{11} \otimes F_1' & S_{12} \otimes F_1' \\ S_{12} \otimes F_2' & S_{22} \otimes F_2' \end{bmatrix}$$

Though $B'S^{-1}BS$ is idempotent, $B'S^{-1}B \begin{bmatrix} S_{11} \otimes F_1'F_1 & S_{12} \otimes F_1'F_2 \\ S_{12} \otimes F_2'F_1 & S_{22} \otimes F_2'F_2 \end{bmatrix}$ is not.

Contradiction, relationship 22) does not hold. Q is not central-χ^2. Q.E.D.

3.3 Alternative Two-pass Estimators for the Segmented Markets Cases

We specify an approximate two-pass linearity statistic and an inefficient covariance matrix estimator for the two-pass "gamma" estimates. These estimators are developed to highlight the differences between the integrated and segmented market cases, and to provide some less computationally intensive prescreening tools for testing the full three-pass or maximum likelihood estimators.

An approximation to the Corollary 1 linearity test is based on the expected value of the random centrality parameter.

$$\tilde{Q} = T\bar{y}'B'S^{-1}B\bar{y} \sim c^2\left(B_r, \bar{m}_Q\right) \qquad 23)$$

$$\bar{m}_Q = TE\left[e'\begin{bmatrix} I_n \otimes F_1\left(F_1'F_1\right)^{-1}G_1 & 0_{nT\,x\,n} \\ 0_{nT\,x\,n} & I_n \otimes F_2\left(F_2'F_2\right)^{-1}G_2 \end{bmatrix} B'S^{-1}B \begin{bmatrix} I_n \otimes G_1\left(F_1'F_1\right)^{-1}F_1' & 0 \\ 0 & I_n \otimes G_2'\left(F_2'F_2\right)^{-1}F_2' \end{bmatrix} e\right]$$

Conditional on B, S, F_1 and F_2, this scalar will equal its trace:

$$\bar{m}_Q = T\,\mathrm{Trace}\left[B'S^{-1}B\begin{bmatrix} S_{11} \otimes G_1'D_{11}^{-1}G_1 & S_{12} \otimes G_1'D_{11}^{-1}D_{12}D_{22}^{-1}G_2 \\ S_{21} \otimes G_2'D_{22}^{-1}D_{21}D_{11}^{-1}G_1 & S_{22} \otimes G_2'D_{22}^{-1}G_2 \end{bmatrix}\right] \qquad 24)$$

An estimate of the covariance matrix of the second-pass gamma estimates followsShanken (1992) and Shanken-Weinstein (1990). From equation 9), which incorporates the errors in the beta estmates:

$$V\left(\bar{R} - E\right) = W + bDb', W = S + \begin{bmatrix} S_{11} \otimes G_1'D_{11}^{-1}G_1 & S_{12} \otimes G_1'D_{11}^{-1}D_{12}D_{22}^{-1}G_2 \\ S_{21} \otimes G_2'D_{22}^{-1}D_{21}D_{11}^{-1}G_1 & S_{22} \otimes G_2'D_{22}^{-1}G_2 \end{bmatrix} \qquad 25)$$

As $\hat{A} = \left(\hat{\bar{X}}_G' \hat{S}^{-1}\hat{\bar{X}}_G'\right)^{-1} \hat{\bar{X}}_G' \; S^{-1}$, $\bar{G} = \hat{A}\bar{R}$, and

$$TV\left(\bar{G} - G\right) = AWA' + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & D_{1_1 1_1} & 0 & D_{1_1 1_2} \\ 0 & 0 & 0 & 0 \\ 0 & D_{1_1 1_2} & 0 & D_{1_2 1_2} \end{bmatrix} \qquad 26)$$

The covariance matrix for all "gammas" is defined analogous to 14).

Direct calculation shows that two-pass covariance estimator 26) does not equal the maximum likelihood covariance matrix 7) or the three-pass estimator 13). Since the two-pass estimator does not attain the m.l.e. lower bound, the estimator is inefficient.

4. A Heteroskedasticity and Autocorrelation Adjustment

Define $Z_t = \left[\hat{X}_{bt} : \hat{X}_{\Gamma t} \right]$, a $2n\ x\ \left[n\sum_{i=1}^{2}\left(k_{1_i}+k_{2_i}\right)+2+\sum_{i=1}^{2}k_{i_i} \right]$ matrix, $t \in [1,T]$. From equation 12), we have the composite parameter vector $q = \left[b', \Gamma' \right]'$. The three-stage least squares "score" matrix is the following:[1]

$$S_t(q) = Z_t'\left(R_{GN_t} - Z_t q\right) \tag{22}$$

We define an autocovariance matrix:

$$Y(j) = \frac{1}{T}\sum_{t=j+1}^{T} S_j(q)' S_j(q) \text{ for } j \geq 0 \tag{23}$$

Following the Newey-West procedure, we aggregate the autocovariance matrices:

$$L = Y(0) + \sum_{j=1}^{p}\left(1 - \frac{j}{p+1}\right)\left(Y(j) + Y(j)'\right) \tag{24}$$

p is the autocorrelation lag truncation parameter. A heteroskedasticity and autocorrelation (of order p) adjusted parameter covariance matrix estimator is the following[2]:

$$V_{HA}(q) = (Z'Z)^{-1} L (Z'Z)^{-1} \tag{25}$$

The associated heteroskedasticity and autocorrelation adjusted Lagrange multiplier test may be developed following Bollerslev and Wooldridge (1992).

5. Conclusion

For more asset groups, we note that the estimators that we have considered generalize. Of course, the error covariance matrix must be nonsingular.

In the diagonal-covariance matrix case, large numbers of securities and groups may be treated. However, the "errors-in-variable" problem of Litzenberger and Ramaswamy (1979) and Shanken (1992) exists in this estimation. Though this extension of our work may be treated in their manner, we leave this exercise and further consideration of Shanken's "n-consistency" in the multi-factor setting to future work.

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1 Davidson and MacKinnon (1993). The Quasi-Maximum Likelihood Method (QMLE) of Bollerslev and Wooldridge (1992) may also be used. In this variant, the score is defined $S_{BW_t}(q) = Z_t' S^{-1}\left(R_{GN_t} - Z_t q\right)$.

2 Following Bollerslev and Wooldridge (1992), a linearized QMLE parameter covariance matrix estimator is $V_{BW}(q) = \left(Z'(S \otimes I_T)Z\right)^{-1} L_{BW} \left(Z'(S \otimes I_T)Z\right)^{-1}$ with both $Y_{BW}(j)$ and F_{BW} corresponding to $S_{BW_t}(q)$.

Lastly, full maximum likelihood estimation (mle) or its robust variants (e.g. QMLE) are clear alternatives to the linearized methods that we treat. Nevertheless, the linearized method will be useful for both determining reasonable parameter starting values and for pre-screening higher dimensional alternatives.

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