Normally Distributed Admissible Choices are Optimal

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Abstract

For both mutually exclusive and portfolio investment choices with normally distributed returns, we show that all elements of the second-order stochastic dominance admissible choice set are Fishburn (1974) optimal. Fishburn's Convex Stochastic Dominance (1974) is used to develop our result. Among normal returns, we conclude that mean-variance admissible choices are optimal. Therefore, Sharpe's (1966) mean-variance ratio is an optimal delegated financial management choice measure. We note that these results also extend to particular Generalized Location Scale distributions of Bawa (1975).

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Normally Distributed Admissible Choices are Optimal

Abstract

For both mutually exclusive and portfolio investment choices with normally distributed returns, we show that all elements of the second-order stochastic dominance admissible choice set are Fishburn (1974) optimal. Fishburn's Convex Stochastic Dominance (1974) is used to develop our result. Among normal returns, we conclude both mean-variance admissible choices are optimal. Therefore, Sharpe's (1966) mean-variance ratio is an optimal delegated financial management choice measure. We note that these results also extend to particular Generalized Location Scale distributions of Bawa (1975).

Stochastic dominance and mean-variance (normal distribution) utility theories emphasize the reduction of an investment choice set to a smaller efficient subset. When investors manifest both non-satiation and risk-aversion, and returns are normally distributed, it is well-known that a simple mean-variance rule identifies inefficient or dominated choice set elements. Any distribution with equal or lower mean and higher variance or a distribution with lower mean and equal variance than another choice distribution will not be chosen [e.g. Bawa (1975).] The remaining undominated choices make up the associated admissible set.

Our question is whether or not normally distributed admissible choice set elements will be chosen by an investor in the second order stochastic dominance (U_2) class, who manifests non-satiation and risk-aversion.[1] In the context of mutually exclusive investment decisions, we resolve an open issue in Meyers (1979), Bawa-Goroff (1982) and Bawa et. al. (1985), and show that all normally distributed admissible choices will be chosen. For completeness, we use Fishburn's Convex Stochastic Dominance construct to prove that portfolios of normally distributed choices are optimal. Independently, Yitzhaki and Mayshar (1997) have also shown this portfolio result. Therefore, both single choice and portfolio choice admissible sets that have

[1] Although we treat the normal distribution case, our results also hold for other Generalized-Location-Scale distributions, including T, stable and log-normal distributions, Bawa (1975).

normal returns are optimal.[2] Furthermore, Sharpe's (1966) classic mean-variance ratio is an optimal investment decision criterion.

The paper is organized as follows. First we provide notation and state definitions. Next, propositions and proofs are stated. Our lemmas are in an appendix.

1. Notation and Definitions

Generally accepted observable behavior has led to the following classes of continuously differentiable utility functions, $u(\cdot)$:

(i) nonsatiation axiom: $u' > 0$

(ii) risk aversion: $u' > 0$, $u'' < 0$

Adopting the notation of Bawa (1975), let the uncertain prospects be characterized by random variables $x_i, i = 1,2,\dots,n+1,$ with known continuous probability distribution functions defined over an open interval R^1 given by (a, b), a < b.

Let the following progressively restrictive set of utility functions, $u(\cdot)$, describe the decision maker's preferences. The utility functions are defined over the space R^1 of realizations of a random variable x:

$$U_1 = \left\{u(x): u(x) \text{ is finite, } u'(x) > 0, \text{ for all } x \in R^1\right\},$$

$$U_2 = \left\{u(x): u(x) \in U_1, u''(x) < 0, \text{ for all } x \in R^1\right\},$$

2 Under CSD, investor utility functions that will choose optimal normally distributed alternatives must exist. In an appendix that is available on request, we also develop these functions, which to our knowledge, have not been presented previously. Fishburn-Vickson (1975, pg. 76, eq. 2.42) provides an expected utility function construction for the case in which the minimum of one discrete point provides admissibility. Given more than a singleton undominated point, multiple utility functions will choose the undominated choice. There are many such functions. Vickson-Brumelle (1975) and Russell-Seo (1989) specify variants of second-order stochastic dominance directly in utility function terms. Building on Russel-Seo (1989), Post (2005) proposes an alternative LPM-based dominating portfolio procedure. Utility function identification for efficient allocations began with Afriat (1967), and were further developed by Varian (1982, 1983), Green-Srivastava (1985, 1986), Dybvig-Ross (1982), and others.

These definitions lead to the following well-known second-order stochastic dominance theorem and definition:[3]

Theorem 1: Second-Order Stochastic Dominance (SSD). For any two cumulative distributions F_i and F_j, F_i is (strictly) preferred to F_j for all utility functions in U_2, if and only if

$$\int_a^x F_i(t)dt \le \int_a^x F_j(t)dt \; \forall\, x \in R \quad (\text{and} < \text{for some } x \in R). \tag{1}$$

Definition: SSD Admissible Set - A subset C of choice set P, its members are not second-order stochastically dominated.

If a choice in P is not in subset C (not admissible), then all investors unanimously drop it from consideration. By dropping these choices, the SSD admissible set substantially reduces the full choice set. In the case of normally distributed choice alternatives, convex second-order dominance is an optimal choice rule.

Definition: Convex Second-Order Stochastic Dominance (CSSD) - A distribution function F_{n+1} is convex second-order stochastically dominated by $\{F_i, i=1,2,\cdots,n\}$, if $\forall\, u \in U_2$, there exists an $F_j \in \{F_1, F_2, \ldots, F_n\}$ such that

$$\int_a^b U(x)dF_j(x) \ge \int_a^b U(x)dF_{n+1}(x). \tag{2}$$

Correspondingly, we introduce the CSSD admissible set.

Definition: CSSD Admissible Set - A subset C of choice set P is CSSD admissible if $\forall\, a \in C$, choice a is not CSSD dominated by any other members of P. Since CSD admissibility is

[3] Assuming only non-satiation, Quirk-Saposnik (1962) and Fishburn (1964) developed First Order Stochastic Dominance. Hadar-Russell (1969, 1971), Hanoch-Levy (1969), and Rothschild-Stiglitz (1970, 1971) provided alternative developments of second-order stochastic dominance. Whitmore (1970) introduced Third-Order Dominance, and Vickson (1975) treated decreasing absolute risk aversion. Porter-Wart-Ferguson (1973), Bawa-Lindenberg-Rafsky (1979) and Aboudi-Thon (1994) provide algorithmic treatments. Levy (1992) provides a review of Stochastic Dominance. Fishburn (1974, 1975) developed Convex Stochastic Dominance (CSD) to

more restrictive than the usual SSD admissibility, the CSSD admissible set is generally smaller than the SSD admissible set.

Let $\lambda = (\lambda_1, \lambda_2 \cdots \lambda_n)$, $\lambda \in \Lambda_n$ with $\lambda_i \geq 0, i=1,2,\cdots n$ and $\sum_{i=1}^{n} \lambda_i = 1$. Following Fishburn (1974), we state the convex generalization of Theorem 1.

Theorem 2: Convex Second-Order Stochastic Dominance (CSSD). F_{n+1} is convex second-order stochastically dominated by $\{F_i, i=1,2,\cdots,n\}$, iff $\exists \lambda \in \Lambda_n$ such that

$$\sum_{i=1}^{n} \lambda_i \int_a^x F_i(t)dt \leq \int_a^x F_{n+1}(t)dt \quad \forall x \in R \text{ (and < for some } x \in R) \tag{3}$$

Conversely, if F_{n+1} is not convex second-order stochastically dominated by $\{F_i, i=1,2,\cdots,n\}$, then it is optimal:

$$\forall \lambda \in \Lambda_n, \exists x \in R, \int_a^x F_{n+1}(t)dt < \sum_{i=1}^{n} \lambda_i \int_a^x F_i(t)dt \Leftrightarrow$$

$$\exists u \in U_2, u(F_i) < u(F_{n+1}), \forall \{i=1,2,\cdots,n\} \tag{4}$$

Therefore, the CSSD admissible, C, is the optimal set.

We also define another important concept relevant to investment choice, the efficiency of a choice set.

Definition: Second-Order Efficient Set - A subset E of choice set P is second-order efficient if it contains the maximizers for all U_2.[4]

Obviously, investors with non-satiation and risk-aversion attributes should only evaluate the minimal second-order efficient choice set in order to make their investment decisions. We show that the minimal efficient choice set is the CSSD admissible set.

identify optimal choices among mutually exclusive alternatives, and the Bawa et. al. (1985) algorithm determinines First-, Second- and Third-Order CSD choices.

4 Bawa and Goroff (1982) demonstrate the equivalence between SD admissibility and efficiency for the portfolio allocation problem.

2. Optimal Choices among Mutually Exclusive Alternatives

Convex Stochastic Dominance (CSD) identifies choice distribution mixtures that dominate other elements of the choice set (the dominated elements). Fishburn (1974) shows that any choice dominated by a mixture of other alternatives will not be chosen. Conversely, any choice that is not so dominated is in the optimal set.

Our method of proof is straightforward. For normal distributions, the appropriate SD decision rule is second-order (SSD). Since normal distributions cross in most cases, first-order stochastic dominance (FSD) is precluded. Under Convex Second-Order Stochastic Dominance (CSSD), we show that the set of mixture distributions necessary to dominate any member of the admissible set is empty. Hence, the admissible set is optimal.

For mutually exclusive choices, the choice space may be written as the following:

$$P = \left\{ \sum_{i=1}^{n} \lambda_i F_i \,\middle|\, \lambda \in \Lambda_n,\ F_i \text{ is normal for } i=1,2,\ldots,n \right\},$$

In Appendix A, we prove two needed Lemmas.

The set of non-SSD dominated distributions (the admissible set) is no smaller than the set of non-CSSD dominated distributions (the optimal set). However, the following theorem shows that in the case of normal distributions, these two concepts coincide. In this case, the two choice sets are identical.

Proposition 1:

Given a set of normal distributions $\Phi = \{F_1, F_2, \cdots, F_n, F_{n+1}\}$, if Φ is a U_2 admissible set, then it is also the CSSD admissible set and optimal.

Proof: Φ is an admissible set; therefore, distributions are mutually undominated. Since in the normal distribution case, SSD is equivalent to the mean-variance decision rule, we can order the distributions in Φ in such a way that

$$\sigma_1 < \sigma_2 < \cdots < \sigma_n \,, \text{ and } \mu_1 < \mu_2 < \cdots < \mu_n \,.$$

The mean and standard deviation of distribution F_{n+1} may be anywhere in the sequence of $F_1, F_2, \cdots F_n$.

Case 1: $\sigma_{n+1} < \sigma_n = \max_{1 \le j \le n} \{\sigma_j\}$. We divide the set Φ in two parts: $\Phi_1 = \{F_1, \cdots, F_k\}$, and $\Phi_2 = \{F_{k+1}, \cdots, F_n\}$, such that $\mu_k < \mu_{n+1} < \mu_{k+1}$ and $\sigma_k < \sigma_{n+1} < \sigma_{k+1}$.

We can take a degenerate distribution as a special case of the normal distribution, by defining its variance to be zero. We replace the set Φ_1 with another set $\hat{\Phi}_1$ such that $\mu(\hat{F}_i) = \mu(F_i)$, $\sigma(\hat{F}_i) = 0$, $i = 1, 2, \cdots k$.

If F_{n+1} cannot be dominated by $\hat{\Phi}_1 \cup \Phi_2$, then F_{n+1} also can't be dominated by $\Phi_1 \cup \Phi_2$ (since each member of Φ_1 is dominated by the corresponding member in $\hat{\Phi}_1$). For members of set Φ_2, we choose a sufficiently small number, r, such that the Variance Dominance Rule can be applied to each element of Φ_2. For simplicity, we keep the notation of F_i, i=1,...,k, instead of $\hat{F}_i$.

From Lemma 1, for any given $\lambda_j > 0$, there exists an r_j such that

$$\int_{-\infty}^{r_j} F_{n+1}(t)dt < \lambda_j \int_{-\infty}^{r_j} F_j(t)dt \,, \quad j = k+1, \cdots n \,. \tag{5}$$

Therefore, there exists a real number $r \in R$, $r < \min\{\mu_i : i=1,\cdots,k,\ r_j : j=k+1,\cdots,n\}$, for any given $\lambda \in \Lambda_n$,

$$\int_{-\infty}^{r} F_{n+1}(t)dt < \sum_{j=k+1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt = \sum_{j=k+1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt + \sum_{j=1}^{k} \lambda_j \int_{-\infty}^{r} F_j(t)dt \tag{6}$$

$$\int_{-\infty}^{r} F_{n+1}(t)dt < \sum_{j=1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt$$

Here, we have used the fact that $\int_{-\infty}^{r} F_j(t)dt = 0$ for $j = 1, \cdots, k$, since $r < \mu_j$.

We have shown that F_{n+1} is not CSSD dominated by $\{F_1, \cdots F_n\}$.

Case 2: $\sigma_{n+1} > \max_{1 \le j \le n} \{\sigma_j\} = \sigma_n$

In this case, from Lemma 1, there exists a sufficiently larger number r_j, such that

$$\int_{r_j}^{+\infty} F_j(t)dt < \int_{r_j}^{+\infty} F_{n+1}(t)dt \quad j = 1,2,\cdots n. \tag{7}$$

Thus,

$$\int_{-\infty}^{r} F_{n+1}(t)dt = 1 - \int_{r}^{+\infty} F_{n+1}(t)dt < 1 - \sum_{j=1}^{n} \lambda_j \int_{r}^{+\infty} F_j(t)dt = \sum_{j=1}^{n} \lambda_j \int_{-\infty}^{r} F_j(t)dt \tag{A-17}$$

where $r > \max_{1 \le j \le n} \{r_j\}$. In this case, we have shown that F_{n+1} can't be CSSD dominated by Φ.

Q.E.D.

3. CSSD Portfolio Choices

For portfolio choices, Baron (1977) has shown that a choice vector, π, dominates the associated mixed strategy, λ_π, for all strictly concave von Neumann-Morgenstern utility functions. We present a corollary to this result as Proposition 2.[5]

[5] Yitzhaki-Mayshar (1997) prove this result in the context of Marginal Stochastic Dominance for general discrete distributions, and provide an alternative derivation for normal distributions. As in our case, there results generalize to a broader class of exchangeable distribution functions. For the general discrete distribution cases,

Our construct is, again, Fishburn's CSSD. Additionally, we need two more lemmas (3 and 4), which are also in Appendix A. Our CSSD efficient portfolio proposition follows:

Proposition 2: The mean-variance efficient portfolio frontier choices are CSSD admissible.

Proof: Given Lemma 4, any mixture of alternatives is dominated by an associated portfolio. Any portfolio not associated with the mean-variance efficient frontier is dominated by some element of the set of portfolios on the efficient frontier. Therefore, mean-variance efficient portfolio choices dominate mixtures of portfolio distributions, and all such portfolios are CSSD admissible.

Like mutually exclusive choice CSSD Proposition 1, Proposition 2 shows that the entire mean-variance efficient portfolio frontier is optimal.

4. Conclusion

For sets of investors with non-satiation and risk-aversion attributes, U_2, who face mutually exclusive normally distibuted investment returns, we have shown that the second-order stochastic dominance (SSD) admissible set is the Bawa et. al. (1985) optimal set and the Bawa-Goroff (1982) strictly best set. By our Fishburn (1974) CSSD methods, or from with the analogous portfolio choice problem results of Yitzhaki-Mayshar (1997), we also know that efficient portfolio choices among normally-distributed alternatives are optimal. Therefore, we conclude that admissible sets of normally distributed choice elements are optimal.

In the absence of mean and variance parameter estimation risk, our results highlight Sharpe's (1966) classic mean-variance ratio as an optimal delegated financial management

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Post (2003), like Yitzhaki-Mayshar, separates dominated and efficient portfolios. For the dominated allocations, Kousmanen (2004) and Bodurtha (2004) provide methods to identify efficient reallocations. In this context, Bodurtha (2005) analyzes alternative priors and the empirical distribution function with discrete approximations to the normal distributions. For utility functions manifesting some risk-seeking preference, Post-Levy (2005) address separation of dominated and efficient portfolio allocations. Goroff-Witt (1982) consider continuous distributions, as well.

choice measure. In this context, a portfolio manager should identify inefficient or dominated choice set elements by this simple mean-variance rule and should not reduce the choice set further before presenting choices to investors.[6]

[6] Though we have noted that our results extend to some other continuous "location-scale" distributions [e.g. Bawa (1975)], Peleg-Yaari (1975) and Peleg (1975), Bawa-Goroff (1982), and Dybvig-Ross (1982) show that the SSD admissible set and various "optimal" sets are not, in general, equal. Further analysis of the respective "risk-aversely efficient" and "regular risk-aversely efficient" random variables, "strictly best choices," and "portfolio efficient sets" is needed. As highlighted by Dybvig-Ross (1982) in the portfolio context and more generally for the mutually exclusive investment choices, the potential for non-convex choice sets raises particular difficulties in this analysis. Alternatively, Bawa-Goroff (1982) have shown that the admissible set is dense in the optimal-strictly best set. Therefore, the delegated manager who provides decision makers with admissible choices is not grossly non-optimal. For first-order Stochastic Dominance, Dybvig (1988) addresses issues.

Appendix - Lemmas

For the mutually exclusive choice case, we now state and prove two lemmas.

Lemma 1: Variance Dominance Rule

Given two distributions F_1 and F_2 with finite variances σ_1^2 and σ_2^2, if we let $\sigma_1 < \sigma_2$, then there exist three numerals x^*, r_1 and r_2, (with $r_1 < r_2$), such that

(I) the density functions $f_1(x)$ and $f_2(x)$ satisfy $f_1(x) < f_2(x)$, if $x < r_1$ or $x > r_2$

(II) the distribution functions have the same value at x^* and satisfy:

$$F_1(x) < F_2(x) \quad \text{if } x < x^* \text{ or } F_1(x) > F_2(x) \quad \text{if } x > x^*.$$

Proof: The proof has three steps.[7]

Step 1: There are exactly two intersection points for $f_1(x)$ and $f_2(x)$.

Therefore, the following equation must have exactly two real roots:

$$\sigma_1 e^{\frac{(x-\mu_1)^2}{2\sigma_1^2}} = \sigma_2 e^{\frac{(x-\mu_2)^2}{2\sigma_2^2}} \tag{A-1}$$

Taking a logarithm of both sides of this equation,and collecting terms, we have

$$0=\left(\sigma_1^2-\sigma_2^2\right)x^2+2x\left(\mu_1\sigma_2^2-\mu_1\sigma_1^2\right)+\left(\sigma_1^2\mu_2^2-\sigma_2^2\mu_1^2-2\sigma_1^2\sigma_2^2\ln\frac{\sigma_1}{\sigma_2}\right) \tag{A-2}$$

We then define the determinant as:

$$\Delta=4\left(\mu_1\sigma_2^2-\mu_2\sigma_1^2\right)^2-4\left(\sigma_1^2-\sigma_2^2\right)\left[\sigma_1^2\mu_2^2-\sigma_2^2\mu_1^2-2\sigma_1^2\sigma_2^2\ln\frac{\sigma_1}{\sigma_2}\right] \tag{A-3}$$

To show that this determinant is greater than zero, we show that the first term on the right-hand side of equation (A-3) is greater than a quantity that is, itself, greater than the second term on the right-hand side of equation (A-3).

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[7] By replacing the mean and variance with the Generalized Location and Scale (ℓ,s) parameters of Bawa (1975, 1979), this proof will show that mean-scale admissible densities within the following classes cross twice: t distributions with the same degree of freedom, Cauchy distributions and log-normal distributions. In these cases, the densities are, like the normal, functions of a standardized random variable, $\left((x-\ell)/s\right)^2$. The differences between and ratios of any two admissible choices for these distributions satisy Lemma 1 (the double crossing property defined in location and scale) and Lemma 2 (the distribution dominance condition.) Though no analytic density functions exist for Stable Distributions other than the normal and Cauchy, the densities associated with stable distributions with the same characteristic exponent and skewness parameter also cross-twice. While the uniform distribution is in the location scale family and admissible uniform distributions are optimal, the switching nature of the mean-scale admissible rule over the range of uniform random variables precludes our line of proof.

Since $\ln\frac{\sigma_1}{\sigma_2}>0$, we need only to show that, $\left(\mu_1\sigma_2^2-\mu_2\sigma_1^2\right)^2\geq\left(\sigma_1^2-\sigma_2^2\right)\left[\sigma_1^2\mu_2^2-\sigma_2^2\mu_1^2\right]$. This inequality is equivalent to $\left(\mu_2-\mu_1\right)^2\geq 0$, so we denote the two real roots as r_1 and r_2.

Step 2: To show (I), we reconsider equation (A-1). Let

$$h(x)\equiv\sigma_1 e^{(x-\mu_1)^2/2\sigma_1^2}-\sigma_2 e^{(x-\mu_2)^2/2\sigma_2^2}. \tag{A-4}$$

Following Step 1, it is straightforward to verify that

$$h'(x)<0 \qquad x\in(-\infty,r_1) \text{ and } h'(x)>0 \qquad x\in(r_2,+\infty) \tag{A-5}$$

Step 3: To show (II), notice that $F_1(\infty)=F_2(\infty)=1$.

Since $F_1(r_1)=\int_{-\infty}^{r_1} f_1(t)\,dt<\int_{-\infty}^{r_1} f_2(t)\,dt=F_2(r_1)$, and $\int_{r_2}^{\infty} f_1(t)\,dt<\int_{r_2}^{\infty} f_2(t)\,dt$,

it must be that
$$\int_{r_1}^{r_2} f_1(t)\,dt>\int_{r_1}^{r_2} f_2(t)\,dt. \tag{A-6}$$

Both $F_1(x)$ and $F_2(x)$ are increasing continuous functions on $(-\infty,\infty)$.[8] Therefore, there exists a unique $x^*\in(r_1,r_2)$, such that

$$F_1\left(x^*\right)=\int_{-\infty}^{x^*} f_1(t)dt=\int_{-\infty}^{x^*} f_2(t)dt=F_2\left(x^*\right), \text{ and } F_1(x)<F_2(x) \text{ if } x<x^*. \tag{A-7}$$

Q.E.D.

The first part of the Variance Dominance Rule states that the density function curve for the smaller variance distribution, F_1, always lies below the other one with larger variance, F_2, on the interval $(-\infty,r_1)$. However, a reversed relationship is true on an interval $(r_2,+\infty)$.

Lemma 2: Given two distribution functions, as in Lemma 1, the value of $F_1(x)$ is negligible compared to the value of $F_2(x)$ if x is sufficiently small.[9]

8 In the log-normal case, $F_1(x)$ and $F_2(x)$ are increasing continuous functions on $(0,\infty)$. The log-normal density crossing points, r_1', r_2', are defined by location-scale and determined in the log space. While the distribution crossing point is a unique $x^{*\prime}\in\left(e^{r_1'},e^{r_2'}\right)$, and distribution dominance follows in the return space.

9 These limits apply for mean-scale admissible t distributions with the same degree of freedom and stable distributions with the same characteristic exponent and skewness. For mean-scale admissible log-normal distributions, equation (15) is defined in location , ℓ ,and scale parameters, s. The necessary SSD log-normal distribution mean condition is imposed with $\ell_2+s_2^2/2>\ell_1+s_1^2/2$. As in the normal case for x, the terms that are quadratic in ln x are the difference in squared scale, which is positive. In this case, the limits are evaluated approaching zero from the right, and all other terms are linear in ln x.

Proof: By L'Hopital's Law, we show that

$$\lim_{x\to-\infty} \frac{F_2(x)}{F_1(x)} = \lim_{x\to-\infty} \frac{F_2'(x)}{F_1'(x)} = +\infty . \tag{A-8}$$

Since $\frac{f_2(x)}{f_1(x)} = \frac{\sigma_1}{\sigma_2} e^{(x-\mu_1)^2/2\sigma_1^2-(x-\mu_2)^2/2\sigma_2^2}$, we show that $\sigma_2^2(x-\mu_1)^2 - \sigma_1^2(x-\mu_2)^2 \to +\infty$.

This is self-evident since $\sigma_1 < \sigma_2$. Similarly, we show that

$$\lim_{x\to+\infty} \frac{f_2(x)}{f_1(x)} = +\infty . \tag{A-9}$$

Q.E.D.

This Lemma is another interpretation of the Variance Dominance Rule, and states that the distribution curve of larger variance not only dominates the distribution curve with a smaller variance, but also that the magnitude of the latter one is actually negligible. In fact as $x\to-\infty$, $F_1(x)$ approaches 0 much faster than $F_2(x)$ does.

For the portfolio choice case, we now state and prove two additional lemmas.

Lemma 3: The SSD integral (1) is convex.

Proof: The SSD integral is a twice continuously differentiable real-valued function on an open interval. Furthermore, its second derivative is the normal density and hence, non-negative throughout its domain. From Rockafellar (1970), convexity follows by Theorem 4.4, and essentially strict convexity follows by Theorem 26.3 (the SSD integral gradient is the normal distribution and is positive over the real line.)

Lemma 4: A portfolio of normally distributed choices SSD dominates the associated mixture of normally distributed choices.

Proof: Given Lemma 3 [convexity of the SSD integral (1)], a convex combination (mixture) of these integrals is no less than the SSD integral defined over the linear combination (portfolio) of the associated random variables.

With integration by parts, we have the following:

$$\int_{-\infty}^{x} F_i(t)dt = \sigma_i \left[\left(\frac{x-\mu_i}{\sigma_i}\right) \Phi\left(\frac{x-\mu_i}{\sigma_i}\right) + \phi\left(\frac{x-\mu_i}{\sigma_i}\right) \right], \text{ and } \Phi\left(\frac{x-\mu_i}{\sigma_i}\right) \text{ and} \tag{A-10}$$

$\phi\left(\frac{x-\mu_i}{\sigma_i}\right)$ are the standard normal distribution and density, respectively.

For a portfolio to CSSD dominate a mixture requires

$$\sigma_p\left[\left(\frac{x-\mu_p}{\sigma_p}\right)\Phi\left(\frac{x-\mu_p}{\sigma_p}\right)+\phi\left(\frac{x-\mu_p}{\sigma_p}\right)\right]\le\alpha\sigma_1\left[\left(\frac{x-\mu_1}{\sigma_1}\right)\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)+\phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right] \qquad \text{(A-11)}$$

$$+(1-\alpha)\sigma_2\left[\left(\frac{x-\mu_2}{\sigma_2}\right)\Phi\left(\frac{x-\mu_2}{\sigma_2}\right)+\phi\left(\frac{x-\mu_2}{\sigma_2}\right)\right],\ \forall x\in(-\infty,\infty) \text{ and } 0<\alpha<1.$$

Defining the portfolio weights to equal the mixture weights, we have

$x_p=\alpha x_1+(1-\alpha)x_2$, $\mu_p=\alpha\mu_1+(1-\alpha)\mu_2$, and

$$\sigma_p^2=\alpha^2\sigma_1^2+2\alpha(1-\alpha)\sigma_1\sigma_2\rho+(1-\alpha)^2\sigma_2^2\neq\left[\alpha\sigma_1+(1-\alpha)\sigma_2\right]^2, \qquad \text{(A-12)}$$

However, setting the correlation equal to one implies that the portfolio standard deviation is a convex combination of the other two standard deviations, and that this standard deviation is an upper bound on the actual portfolio standard deviation:

$$\sigma_p\le\sigma_{p|\rho=1}=\alpha\sigma_1+(1-\alpha)\sigma_2 \qquad \text{(A-13)}$$

Therefore,

$$\sigma_p\left[\left(\frac{x-\mu_p}{\sigma_p}\right)\Phi\left(\frac{x-\mu_p}{\sigma_p}\right)+\phi\left(\frac{x-\mu_p}{\sigma_p}\right)\right]\le\sigma_{p|\rho=1}\left[\left(\frac{x-\mu_p}{\sigma_{p|\rho=1}}\right)\Phi\left(\frac{x-\mu_p}{\sigma_{p|\rho=1}}\right)+\phi\left(\frac{x-\mu_p}{\sigma_{p|\rho=1}}\right)\right] \qquad \text{(A-15)}$$

$$\le\alpha\sigma_1\left[\left(\frac{x-\mu_1}{\sigma_1}\right)\Phi\left(\frac{x-\mu_1}{\sigma_1}\right)+\phi\left(\frac{x-\mu_1}{\sigma_1}\right)\right]+(1-\alpha)\sigma_2\left[\left(\frac{x-\mu_2}{\sigma_2}\right)\Phi\left(\frac{x-\mu_2}{\sigma_2}\right)+\phi\left(\frac{x-\mu_2}{\sigma_2}\right)\right],$$

$$\forall x\in(-\infty,\infty) \text{ and } 0<\alpha<1.$$

Q.E.D.

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