An Asset Allocation Puzzle – Prior Perspective and Posterior Resolution

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Abstract

Based in the linear inequalities of Yitzhaki-Mayshar (1997), we develop a linear program to identify dominated and efficient portfolio choices. We extend this method to accommodate discrete approximations to priors of distributions and processes other than the empirical distribution. Based in Ferguson (1974) and Bawa-Brown-Klein (1979), we illustrate our nonparametric Bayesian method to resolve a controversy regarding investment allocations considered by Canner-Mankiw-Weil (1997) and Shalit-Yitzhaki (2004).
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Based in the linear inequalities of Yitzhaki-Mayshar (1997), we develop a linear program to identify dominated and efficient portfolio choices. We extend this method to accommodate discrete approximations to priors of distributions and processes other than the empirical distribution. Based in Ferguson (1974) and Bawa-Brown-Klein (1979), we illustrate our nonparametric Bayesian method to resolve a controversy regarding investment allocations considered by Canner-Mankiw-Weil (1997) and Shalit-Yitzhaki (2004).

Portfolio theory has, generally, followed two tracks. In one track, a particular continuous multivariate returns probability distribution is assumed, and the investment allocation is optimized. Though the Stochastic Dominance paradigm includes this first track, stochastic dominance applications have tended to work with general discrete and finite multinomial empirical distributions. An article and comment in the American Economic Review each take one of these tracks, and conclude with diametrically opposed results.

The particular application is an efficiency evaluation of Wall Street analysts’ equity, bond and cash portfolio allocations. Canner, Mankiw and Weil (1997) assume multivariate normality, and find that all ten analysts’ allocation suggestions are dominated. Shalit-Yitzhaki (2004) assume the discrete multinomial empirical distribution, and find that all of the ten analysts’ allocation suggestions are undominated and efficient. Clearly, the prior distribution assumptions of the authors’ of these works differ, and it is not surprising that they reach somewhat different conclusions. However, the degree of their differences is remarkable.

To resolve the difference, we begin with one linear programming asset allocation specification of Yitzhaki-Mayshar (YM-1997.) The implementation is equivalent to Post (2003), with some changes that are identified in Bodurtha (2004). These YM and Post tests
both start with a portfolio allocation, and determine if it may be improved. If so, then the initial allocation is dominated.

Though our linear program is specified for discrete returns distributions, continuous distributions are readily approximated with sufficient discrete approximations. Our appendix specifies our approximation method.

In accordance with the Canner, Mankiw and Weil (1997) result, running this linear program over a discrete multinomial approximation to the multivariate normal distribution finds all ten analysts’ allocations to be dominated. Of course, this same linear program specification replicates the Shalit-Yitzhaki (2004) empirical multinomial distribution result, and finds all allocations to be undominated and efficient.

As originally developed in Ferguson (1973, 1974), a Bayesian perspective provides resolution to this apparent paradox. Bawa-Brown-Klein (1979) specifically treat portfolio allocation estimation risk in the Ferguson context. In this framework, a discrete Dirichlet Process Prior Distribution is set at the decision time, and subsequent return realizations augment the prior. Since the Dirichlet Process Prior is conjugate to the multinomial, the posterior distribution is very tractable. A mixing parameter of the prior and observed multinomial distribution sets the relative weight of the prior and realized returns in the resulting posterior distribution.

For the asset allocation problem at hand, we set the decision time as the end of the Canner-Weil-Mankiw and Shalit-Yitzhaki December 1926-December 1992 annual observation period. The sample data is used to form the normal prior, as well as the mixed normal and general empirical distribution prior. The posterior on December 31, 2003 is the mixture of the prior distribution and the empirical distribution function of the 11 new annual
To resolve the Canner-Mankiw-Weil and Shalit-Yitzhaki findings, we have two mixing parameters with which to adapt the posterior.

We identify the breakeven mixture levels such that some risk-averse investor is at her margin in choosing the analyst allocation. Such an investor will choose the allocation if the prior moves toward the general empirical distribution prior, and will reject the allocation should the prior move toward the normal distribution. As an implementation of Ferguson’s (1974) Bayesian Analysis, the method should also be useful in the risk management and investment performance evaluation contexts.

1. **Yitzhaki-Mayshar Second-order stochastic Dominance Portfolio Specification**

Second-order stochastic dominance (SSD) continuously differentiable utility functions are specified: \( u(.) \in U_2: u' > 0, u'' < 0 \). The utility functions are defined over the n-dimensional Euclidean space and map into a non-empty, closed and convex subset, \( P \). For portfolio problems, the utility function is defined over the feasible portfolio set: the positive orthant of the Euclidean space. There are \( N \) investment choices represented by return vectors, \( x \in \mathbb{R}^N \). Investment allocations, \( \lambda \), over investment returns are positive and sum to one.

To identify dominated alternatives, Yitzhaki-Mayshar (YM-1997) treat discrete empirical distributions. These multinomial distributions are ordered by the returns on the portfolio being evaluated, \( \tau, (\Theta \equiv \{1, \ldots, t, \ldots, T\}) \).

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1 Bawa (1980) extends the Ferguson result beyond the risk-averse (quadratic loss) case.

2 For Shalit-Yitzhaki (2004), the posterior is a mixture of the diffuse Dirichlet Process Prior and the empirical distribution. This posterior is the full sample empirical distribution. For Canner-Mankiw-Weil (1997), the prior is taken to be normal. Alternatively, their prior could be Student-T. In this case, the posterior would also be distributed T, but with more degrees of freedom. Both Normal and Student-T distributions are closely related location-scale distributions, and all choice distributions have the same degrees of freedom. Though the particular choices for a given utility function will differ across the distributions, the efficient sets are equivalent and may be expressed in mean and variance. See Bawa-Brown-Klein (1979)
For a particular portfolio allocation, \(\tau\), to be efficient, the allocation must be a maximizer for some second-order stochastic dominance utility function. Developed from Shalit-Yitzhaki (1994) Marginal Conditional SSD, the YM specification (Prop. 7 – eq. 2) is defined in terms of Absolute Concentration Curves (ACC):

\[
\text{ACC}_{it} = C_{it} = C_{it-1} + \pi_i x_{it}
\]

The period \(t\) return is multiplied by the probability of the outcome, \(\pi_t\). YM identify a dominated alternative if the following system of constraints has a solution for at least one non-zero delta, \(\delta_i\):

\[
\sum_{i=1}^{N} \delta_i \leq 0, \quad \delta_i = \lambda_i - \tau_i \quad 2)
\]

\[
\sum_{i=1}^{N} \delta_i c_{it} \geq 0 \quad \forall t = 1, \ldots, T
\]

As our initial specification of a linear program that will determine the feasibility of this constraint set, we specify:

\[
\begin{align*}
\text{Max} & \quad \sum_{i=1}^{N} \left( \delta_i^+ - \delta_i^- \right) c_{it} \\
\text{s.t.} & \quad \sum_{i=1}^{N} \left( \delta_i^+ - \delta_i^- \right) c_{it} \geq 0 \quad \forall t = 1, \ldots, T - 1 \\
& \quad \sum_{i=1}^{N} \delta_i^+ - \delta_i^- = 0 \\
& \quad \delta_1^+, \ldots, \delta_N^+, \delta_1^-, \ldots, \delta_N^- \geq 0 
\end{align*}
\]

Unless the optimand of this LP is greater than zero, the portfolio allocation, \(\tau\), that was used to order the multinomial return states, \(\Theta\), is dominated. Since each delta difference term is unbounded, efficient portfolios result in unbounded (negative) solutions.

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3 The corresponding LP with an optimand that is the minimum feasible sum of the delta investment rates is equivalent to this Max at the associated Min. Also note that any of the ACC-related constraints may serve as
We change the optimization variable from marginal allocations, $\delta_i$, to investment allocations, $\lambda_i$. The initial evaluated portfolio allocation is $\tau_i$. The sums of both investment weights ($\lambda_i$ and $\tau_i$) are one, and we have the following matrix program definition:

$$
\begin{align*}
\max_{\{\lambda_1, \ldots, \lambda_N\}} & \quad (c'_{T} - e'c_{\tau_{T}})\lambda \\
\text{s.t.} & \quad \lambda \leq \\
& \quad -c'_{\tau_{T-1}} \leq \lambda \leq -c'_{\tau_{1}} \\
& \quad e'\lambda \leq 1 \text{ and } -e'\lambda \leq -1 \quad (\text{or } e'\lambda = 1) \\
& \quad c'_i = [c_{1i}, \ldots, c_{Ni}] 
\end{align*}
$$

4) From dual linear programming theory, we have a more efficient alternative for calculation:

$$
\min_{\{y=[y_1, y_2, b_1, b_{T-1}]\}} [1, -1, -c_{\tau_{1}}, \ldots, -c_{\tau_{T-1}}]y \\
\text{s.t.} \quad [e, -e, c_{1}, \ldots, c_{T-1}]y \leq \\
& \quad [c_{IT} \vdots c_{NT}]
$$

5) 2. Multinomial and Dirichlet Process Prior Distributions

The discrete support points of the distribution domain include three interspersed sets: historical, model, and update. For our application, the historical set, $E$, supports the empirical prior of annual observations from December 1926-December 1992. Within this set, there are $E$ outcomes and each is equally likely. This set is associated with the Shalit-Yitzhaki (2004) prior.

These historical observations are also used to form the model, $\phi$, multivariate normal distribution set of outcomes. As developed in our Appendix, this set is dense relative to the optimand. Choice of a particular ACC-related constraint is linked to the utility function risk aversion that
empirical prior outcome space. Nevertheless, each of the $\phi$ outcomes is unique, and the probability of each outcome within the set is equal. This set is associated with the Canner-Mankiw-Weil (1997) prior.

The update distribution is formed with annual return realizations from December 1993-December 2003. There are $n$ of these unique outcomes, and the outcome probabilities are equal. This set is the updating sample that takes the prior to the posterior.

Given the discrete multinomial nature of the historical and update distribution, and our construction of a multinomial distribution approximation to the normal distribution prior, a natural prior over which to form our posterior follows Ferguson (1973, 1974), the Dirichlet Process Prior. The key feature of this prior is that its multinomial characteristics carry through to the posterior.\(^4\)

We have three weight parameters for constructing the posterior, the weight for the multinomial empirical prior distribution, $w_E$, and the weight for the multivariate normal prior distribution, $w_\phi$. The weight for the update observations is one.

$$
\pi_E = w_E / (n + E w_E + \phi w_\phi), \quad \pi_N = w_N / (n + E w_E + \phi w_\phi),
$$

$$
\pi_U = 1 / (n + E w_E + N w_N),
$$

The combined measure of the two priors is $\alpha(R)$, and equals $E w_E + \phi w_\phi$. The total measure of the outcome space is $E w_E + \phi w_\phi + n$, or $\alpha(R) + n$. The relative weights set the significance of the particular prior in the posterior. However, we use these weights only as a

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\(^4\) Blackwell (1973) showed that this posterior must be discrete. Nevertheless, the potential outcome space fills the rationals and is countably infinite. See Seaman-Richardson (2004) for related discussion of prior probability properties as they approach the limit. Given the inherent discreteness of money prices and, hence, returns, this structure is quite tenable. For continuous supporting prices, alternatives are Polya tree non-parametric approach (O’Hagan-Forster, 2004, 13.37-13.39), or a parametric approach which treats the grouping of outcomes at the reported points.
mechanism to define the Dirichlet Process Prior probability parameters. These probabilities, and not the weights, specify the prior parameters.

**Ferguson (1974) Theorem 1**: If $F$ is the Dirichlet Process Distribution with parameter vector $\alpha$, and if $x_1, \ldots, x_n$ is a sample from $F$, then the posterior distribution of $F$ given $x_1, \ldots, x_n$ is a Dirichlet Process Distribution with parameter vector $\alpha + \sum_{i=1}^{n} \delta_{x_i}$, where $\delta_x$ is the measure giving mass one to $x$.

**Example 1**: In distribution function form, the posterior parameter of the process is $\alpha(t) + \sum_{i=1}^{n} I_{[x_i, \infty)}(t)$, and $I_{[x_i, \infty)}(t)$ is the indicator function. With no observations on $F$, the Bayes estimate is

$$\hat{F}_n(t) = E\left(F(t) \mid x_1, \ldots, x_n\right) = \alpha(t)/\alpha(R) = F_0(t)$$

The sample-based Bayes estimate is the following:

$$\hat{F}_n(t) = E\left(F(t) \mid x_1, \ldots, x_n\right) = \left(\alpha(t) + \sum_{i=1}^{n} I_{[x_i, \infty)}(t)\right)/\left(\alpha(R) + n\right)$$

$$= p_n F_0(t) + (1-p_n)$$

$$F_n(t) = \frac{1}{n} \sum_{i=1}^{n} I_{[x_i, \infty)}(t), \quad p_n = \alpha(R)/\left(\alpha(R) + n\right)$$

**Example 2**: The Bayes estimate of the mean of an unknown distribution, $F$, is defined for the loss function $L(F, \mu) = \left(\int t^2 dF(t) - \mu^2\right)^2$. For the Dirichlet Process Prior distribution with $\int t^2 d\alpha(t) < \infty$ and based on the sample $x_1, \ldots, x_n$ from $F$, the posterior mean is the following:

$$\hat{\mu}_n = E\left(\int t dF(t) \mid x_1, \ldots, x_n\right) = \int t dE\left(F(t) \mid x_1, \ldots, x_n\right)$$

$$= \int t d\hat{F}_n(t) = p_n \mu_0 + (1-p_n) \bar{X}_n$$

where $\bar{X}_n$ is the sample mean, and $\mu_0$ is the prior mean guess,

$$\mu_0 = \int t dF_0(t)$$

As the equation 1) ACC are incremental means, the associated posterior estimators follow from Example 2. Particularly, the posterior mean is the posterior ACC$_T$. 

-7-
Given the difference in the Canner-Mankiw-Weil (1997) and Shalit-Yitzhaki (2004) results, there are two sets of prior weights and associated posteriors that will be examined. The first simply takes the Canner-Mankiw-Weil prior and mixes it with the subsequent n return observations. We may identify a breakeven prior weight set \((w_E=0, w_N=b_N)\) such that the analyst allocations are not dominated.

The second set distinguishes the two prior assumptions. We give unit weight to all empirical observations, \(n + E\). Then, we search for a breakeven weight on the normal portion of a mixed empirical multinomial and multivariate normal prior distribution \((w_E=1, w_N=b_{NE})\) such that the analyst allocations are not dominated. For any prior weight greater than this breakeven, the allocation will be dominated, and is consistent with the Canner-Mankiw-Weil prior. For lower prior weights, the allocation is efficient, and is consistent with the Shalit-Yitzhaki prior.

3. Empirical Results

The following Table reports our findings.

<table>
<thead>
<tr>
<th>Analyst</th>
<th>Allocation Weights</th>
<th>Breakeven Prior Weights</th>
<th>Normal (w_E=0, w_N=b_N)</th>
<th>Mixed Empirical &amp; Normal (w_E=1, w_N=b_{NE})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>{0.50,0.30,0.20}</td>
<td>0.00525</td>
<td>0.0059</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>{0.20,0.40,0.40}</td>
<td>0.02</td>
<td>0.0225</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>{0.20,0.35,0.45}</td>
<td>0.035</td>
<td>0.036</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>{0.10,0.40,0.50}</td>
<td>0.0375</td>
<td>0.035</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>{0.05,0.40,0.55}</td>
<td>0.05</td>
<td>0.0525</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>{0.10,0.30,0.60}</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>{0.05,0.30,0.65}</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>{0.05,0.20,0.75}</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>{0.00,0.20,0.80}</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>{0.00,0.00,1.00}</td>
<td>*</td>
<td>*</td>
<td></td>
</tr>
</tbody>
</table>

* With the 1993-2003 annual return updates to the posterior, the tested allocation is efficient.
The most important result reported is that the posterior update with 1993-2003 returns has allocations with more than 60% in the S&P500 equities being undominated. We find this finding remarkable because these eleven observations overcome the dominance of the allocations for the 4096 observations in the discrete approximation to the normal distribution.\(^5\) We emphasize that the normal discrete approximation rejects efficiency for all allocations without this additional data. Constructively, these additional 11 points impose effective dominance constraints that cannot be satisfied up to the limit of the prior weight into the normal distribution prior.

For the allocations with less than 60% invested in the S&P 500 portfolio, we identify breakeven prior weights. For the first analyst, 50% is invested in one-year maturity Treasuries, 30% is invested in corporate bonds and 20% is invested in the S&P500 index portfolio. And, there exists an investor in \(U_2\) with this 0.00525 normal prior weight for equation 6) for whom the allocation \{50-30-20\} is efficient. Giving more prior weight to the normal distribution causes the Analyst’s allocation to be dominated for some investor in \(U_2\). For the normal prior Table column, Analyst allocations 2 –5 are interpreted analogously.

The last column of the Table implements a mixed prior of the empirical distribution of Shalit-Yitzhaki (2004) and the normal prior of Canner-Mankiw-Weil (1997). The empirical prior is given a unit weight, like the updating posterior observations, and normal prior distribution weight varies.

Like the normal prior case, the last five Analyst allocations (corresponding to greater than 55% S&P500 allocation) are all efficient. Also as in the normal case, the first five

\(^5\) Of course, the late 1990’s were an extraordinary period. It has been emphasized that empirical Stochastic Dominance tests are sensitive to tail probabilities. For our aim of providing some resolution to the divergent results of Canner-Mankiw-Weil and Shalit-Yitzhaki, this issue is secondary. Nevertheless, significance testing of our results may also follow Beach-Davidson (1983), Nelson-Pope (1991), Dardanoni-Forcina
Analyst allocations are dominated if sufficient weight is given to the normal component of the mixed prior. In all but one of these mixed prior cases, the weight that must be given to the normal prior is higher in order to overcome the efficiency manifest in the full empirical distribution prior. However, this result is not general because the 4th Analyst allocation requires a lower prior weight on the normal prior mixed with the empirical distribution than did only the normal prior.6

4. Conclusion

Our main objective is to reconcile seemingly incompatible portfolio efficiency test results. In the first case, Canner-Mankiw-Weil (1997) found that ten investment analyst allocations across cash, corporate bond and equity portfolios were dominated. Their distributional assumption was multivariate normal. Analyzing the same data, but making the general empirical distribution assumption, Shali-Yitzhaki (2004) find the exact opposite result: All allocations are efficient.

In a Bayesian context, we identify the authors’ distributional assumptions with their respective priors. Then, based in the Ferguson (1973, 1974) multinomial and Dirichlet Process Prior distribution specification, we examine portfolio performance on the mixed posterior distribution.


6 Bodurtha (2004) provides a tractable procedure to solve for the efficient allocation of the dominated allocations. Kuosmanen (2004) provides an alternative method, but we did not use it due to its computational resource intensity. The following efficient reallocations result in the listed average return gains.

<table>
<thead>
<tr>
<th>Initial Allocation {Cash, Bond, Equity}</th>
<th>Efficient Allocation {Cash, Bond, Equity}</th>
<th>Average Return Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>{0.50, 0.30, 0.20}</td>
<td>{0.0, 0.0, 1.0}</td>
<td>3.5%</td>
</tr>
<tr>
<td>{0.20, 0.40, 0.40}</td>
<td>{0.35, 0.20, 0.45}</td>
<td>0.12%</td>
</tr>
<tr>
<td>{0.20, 0.35, 0.45}</td>
<td>{0.29, 0.23, 0.48}</td>
<td>0.04%</td>
</tr>
<tr>
<td>{0.10, 0.40, 0.50}</td>
<td>{0.20, 0.27, 0.53}</td>
<td>0.04%</td>
</tr>
<tr>
<td>{0.05, 0.45, 0.55}</td>
<td>{0.12, 0.31, 0.57}</td>
<td>0.02%</td>
</tr>
</tbody>
</table>
Since the authors’ sample period, 1926-1992, 11 new annual return observations have been realized. We find that the impact of these observations on the posterior is strong. For any prior, all five analyst allocations that have at least 60% invested in the S&P 500 portfolio are efficient. This result holds even giving the Normal prior a maximal weight.\(^7\) Therefore, a first partial resolution to the divergent efficiency test results of the Canner-Mankiw-Weil and Shalit-Yitzhaki works is that these five allocations are efficient for the updated and full 1926-2003 annual return history.

Nevertheless, the first five Analyst allocations are still dominated if sufficient weight is assigned to the normal prior. Therefore, these portfolio results continue to support the allocation inefficiency finding of Canner-Mankiw-Weil (1997). At the same time, if more weight is given to the updating observations and/or the combined updating observations and the empirical distribution prior, then the allocation efficiency finding of Shalit-Yitzhaki (2004) is supported.

Therefore, our resolution of this asset allocation puzzle is found in alternative priors. The proposed test is well specified for other efficiency tests in which the implemented data generating process is multinomial or from multivariate Monte Carlo Markov Chains. A large number of multperiod fixed income and derivative pricing models fall in this category. Importantly, more general distributions than the multivariate normal may be treated in our manner.

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\(^7\) Weights in the trillion trillion don’t result in dominance. Of course, the limit of raising the normal prior weight will result in dominance because our discrete multinomial approximation to the normal yields dominance. This result follows the Canner-Mankiw-Weil (1997) result.
Appendix – Multinomial Approximation to the Multivariate Normal Distribution

There are many discrete approximation possibilities for the multivariate normal distribution. Natural choices are Quasi-Monte Carlo, e.g. Boyle-Joy-Tan (1994), stochastic mesh Monte Carlo, e.g. Broadie-Glasserman 2004), and multinomial, e.g. Heath-Jarrow-Morton (1992) and Amin-Bodurtha (1995). Though in applications on a digital computer these methods will all generate discrete multivariate distributions, the form of the Ferguson Bayesian approach is most consistent with the Multinomial. The other methods require inversion of the normal distribution, and, hence, the underlying distribution is continuous. The multinomial may also converge to the continuous distribution, but it may be readily fixed to the discrete case based on appropriate discreteness of prices and, hence, rationality of returns.

Amin-Bodurtha (1995) specify a multinomial process over unequal time steps that is path dependent and converges to the multivariate normal. They implement this process both for a more complicated underlying case, cross-country interest rates and foreign exchange, and for distributions more general than the normal.

To generate multiple random variable processes, we must permit appropriate correlations among the draws. A simple way is to draw uncorrelated variables and transform the variables appropriately. The necessary transformation involves the covariance matrix. Consider three uncorrelated (under normality also independent random variables) with means zero and variance one, \((Y_1, Y_2, Y_3)\):
Probability Distribution in terms of \( (Y_1, Y_2, Y_3) \):

<table>
<thead>
<tr>
<th>State</th>
<th>( (Y_1(t), Y_2(t), Y_3(t)) )</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>((1,1,1))</td>
<td>1/4</td>
</tr>
<tr>
<td>B</td>
<td>((1,-1,-1))</td>
<td>1/4</td>
</tr>
<tr>
<td>C</td>
<td>((-1,1,-1))</td>
<td>1/4</td>
</tr>
<tr>
<td>D</td>
<td>((-1,-1,1))</td>
<td>1/4</td>
</tr>
</tbody>
</table>

We create three correlated standard (mean zero and variance one) random variables \((X_1, X_2, X_3)\):

\[
X_1 = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} Y_1
\]

\[
X_2 = \begin{bmatrix} c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix} Y_2 = X = CY, \quad C = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{21} & c_{22} & c_{32} \\ c_{31} & c_{32} & c_{33} \end{bmatrix}
\]

The correlation matrix of the correlated random variables, \( \Gamma \), is defined as the outer product matrix decomposition: \( \Gamma = CC' \).\(^8\) From the correlated zero mean and unit variance variables, we have the multinomial price changes:

\[
S_i(t+h) = S_i(t) + \alpha_i(t,.)h + \sigma_i(t,.)X_i(t)\sqrt{h}
\]

\[
S_2(t+h) = S_2(t) + \alpha_2(t,.)h + \sigma_2(t,.)X_2(t)\sqrt{h}
\]

\[
S_3(t+h) = S_3(t) + \alpha_3(t,.)h + \sigma_3(t,.)X_3(t)\sqrt{h}
\]

\(S_i(t+h)\) is the time \( t \) plus interval \( h \) spot price of asset \( i \), \( \alpha_i(t,.) \) is the time \( t \) drift function, and \( \sigma_i(t,.) \) is the time \( t \) standard deviation function.

A key component of this procedure is the use of unequal time steps to generate uniform coverage over the domain. This construct generates path-dependent spot price.

---

\(^8\) Amin-Bodurtha (1995) decompose the covariance matrix with the lower diagonal Cholesky decomposition matrix. The cross-variable fill of the multivariate space is much more uniform with the revised decomposition. The previous specification is preferable for some derivative pricing applications and complicated covariance matrix structures because the Cholesky decomposition is analytic.
evolution. From simple measurements of deviations of multinomial coverage relative to the normal, the following time steps have been used in a six, \( H \), step approximation:\(^9\)

\[ h_1 = 1/63, \ h_2 = 2/63, \ h_3 = 4/63, \ h_4 = 8/63, \ h_5 = 16/63, \ h_6 = 32/63 \]

The annualized means and covariances of the cash, corp. bond and S&P 500 index returns are the following:

<table>
<thead>
<tr>
<th>1926-1992 Annual Returns</th>
<th>Covariance Matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
</tr>
<tr>
<td>Cash</td>
<td>0.006</td>
</tr>
<tr>
<td>Corp. Bond</td>
<td>0.021</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>0.090</td>
</tr>
</tbody>
</table>

The following plot depicts the discrete approximations to the normal:

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\(^9\) Amin-Bodurtha (1995) suggest the following recursion with weights between two and three:

\[ h_i = d_i / \sum_{i=1}^{H} d_i \]. We use the weight of two.
With regard to coverage of the conditional dimensions, the following figures plot the joint coverage across the three co-spaces. The six cluster plots in each figure correspond to the sequence of each step to the approximation. We observe relatively uniform coverage:\(^{10}\)

![Graphs showing joint coverage across three co-spaces](image)

In the Amin-Bodurtha (1995) application, the time steps are allowed to go to the limit to generate approximations to continuous diffusions. In the limit, Donsker’s Theorem implies that our multinomials go to multivariate normal. As noted in the main body of the paper, our specification of the Dirichlet process prior requires that we not go through to the limit.\(^{11}\)

As the intervals in our approximation shorten, at some point, multiple realizations of a return outcome must occur. In practice and, particularly for our six step and 4096 state

\(^{10}\) The diagonal presentation of the clusters have no significance and only spread out the clusters for each time step.

\(^{11}\) Seaman-Richardson (2004) address this issue in a different context.
approximation, no state is generated multiple times. Higher dimensional orders are handled analogously.
References


